

Nonlinear Dynamics of a System of Particle-Like Wavepackets

Anatoli Babin

University of California at Irvine
USA

Alexander Figotin

University of California at Irvine
USA

This work continues our studies of nonlinear evolution of a system of wavepackets. We study a wave propagation governed by a nonlinear system of hyperbolic PDE's with constant coefficients with the initial data being a multi-wavepacket. By definition, a general wavepacket has a well-defined principal wave vector, and, as we proved in previous works, the nonlinear dynamics preserves systems of wavepackets and their principal wave vectors. Here we study the nonlinear evolution of a special class of wavepackets, namely particle-like wavepackets. A particle-like wavepacket is of a dual nature: on one hand, it is a wave with a well-defined principal wave vector, on the other hand, it is a particle in the sense that it can be assigned a well-defined position in the space. We prove that under the nonlinear evolution a generic multi-particle wavepacket remains to be a multi-particle wavepacket with high accuracy, and every constituting single particle-like wavepacket not only preserves its principal wave number but also it has a well-defined space position evolving with a constant velocity which is its group

velocity. Remarkably the described properties hold though the involved single particle-like wavepackets undergo nonlinear interactions and multiple collisions in the space. We also prove that if principal wavevectors of multi-particle wavepacket are generic, the result of nonlinear interactions between different wavepackets is small and the approximate linear superposition principle holds uniformly with respect to the initial spatial positions of wavepackets. Bibliography: 41 titles.

1. Introduction

The principal object of our studies here is a general nonlinear evolutionary system which describes wave propagation in homogeneous media governed by hyperbolic PDE's in \mathbb{R}^d , $d = 1, 2, 3, \dots$, is the space dimension, of the form

$$\partial_\tau \mathbf{U} = -\frac{i}{\varrho} \mathbf{L}(-i\nabla) \mathbf{U} + \mathbf{F}(\mathbf{U}), \quad \mathbf{U}(\mathbf{r}, \tau)|_{\tau=0} = \mathbf{h}(\mathbf{r}), \quad \mathbf{r} \in \mathbb{R}^d, \quad (1.1)$$

where (i) $\mathbf{U} = \mathbf{U}(\mathbf{r}, \tau)$, $\mathbf{r} \in \mathbb{R}^d$, $\mathbf{U} \in \mathbb{C}^{2J}$ is a $2J$ -dimensional vector; (ii) $\mathbf{L}(-i\nabla)$ is a linear selfadjoint differential (pseudodifferential) operator with constant coefficients with the symbol $\mathbf{L}(\mathbf{k})$, which is a Hermitian $2J \times 2J$ matrix; (iii) \mathbf{F} is a general polynomial nonlinearity; (iv) $\varrho > 0$ is a *small parameter*. The properties of the linear part are described in terms of dispersion relations $\omega_n(\mathbf{k})$ (eigenvalues of the matrix $\mathbf{L}(\mathbf{k})$). The form of the equation suggests that the processes described by it involve two time scales. Since the nonlinearity $\mathbf{F}(\mathbf{U})$ is of order one, nonlinear effects occur at times τ of order one, whereas the natural time scale of linear effects, governed by the operator \mathbf{L} with the coefficient $1/\varrho$, is of order ϱ . Consequently, the small parameter ϱ measures the ratio of the slow (nonlinear effects) time scale and the fast (linear effects) time scale. A typical example of an equation of the form (1.1) is the nonlinear Schrödinger equation (NLS) or a system of NLS's. Many more examples including a general nonlinear wave equation and the Maxwell equations in periodic media truncated to a finite number of bands are considered in [7, 8].

As in our previous works [7, 8], we consider here the nonlinear evolutionary system (1.1) with the initial data $\mathbf{h}(\mathbf{r})$ being the sum of wavepackets. *The special focus of this paper is particle-like localized wavepackets which can be viewed as quasiparticles.* Recall that a general wavepacket is defined as such a function $\mathbf{h}(\mathbf{r})$ that its Fourier transform $\hat{\mathbf{h}}(\mathbf{k})$ is localized in a β -neighborhood of a single wavevector \mathbf{k}_* , called *principal wavevector*, where β is a *small parameter*. The simplest example of a wavepacket is a function

of the form

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \beta^{-d} e^{-i\mathbf{k}\mathbf{r}_*} \hat{h}\left(\frac{\mathbf{k} - \mathbf{k}_*}{\beta}\right) \mathbf{g}_n(\mathbf{k}_*), \quad \mathbf{k} \in \mathbb{R}^d, \quad (1.2)$$

where $\mathbf{g}_n(\mathbf{k}_*)$ is an eigenvector of the matrix $\mathbf{L}(\mathbf{k}_*)$ and $\hat{h}(\mathbf{k})$ is a scalar Schwarz function (i.e., it is an infinitely smooth and rapidly decaying one). Note that for $\hat{\mathbf{h}}(\beta, \mathbf{k})$ of the form (1.2) we have its inverse Fourier transform

$$\mathbf{h}(\beta; \mathbf{r}) = h(\beta(\mathbf{r} - \mathbf{r}_*)) e^{i\mathbf{k}_*(\mathbf{r} - \mathbf{r}_*)} \mathbf{g}_n(\mathbf{k}_*), \quad \mathbf{r} \in \mathbb{R}^d. \quad (1.3)$$

Evidently, $\mathbf{h}(\beta, \mathbf{r})$ described by the above formula is a plane wave $e^{i\mathbf{k}_*\mathbf{r}} \mathbf{g}_n(\mathbf{k}_*)$ modulated by a slowly varying amplitude $h(\beta(\mathbf{r} - \mathbf{r}_*))$ obtained from $h(\mathbf{z})$ by a spatial shift along the vector \mathbf{r}_* with a subsequent dilation with a large factor $1/\beta$. Clearly, the resulting amplitude has a typical spatial extension proportional to β^{-1} and the spatial shift produces a noticeable effect if $|\mathbf{r}_*| \gg \beta^{-1}$. The spatial form of the wavepacket (1.3) naturally allows us to interpret $\mathbf{r}_* \in \mathbb{R}^d$ as its position and, consequently, to consider the wavepacket as a particle-like one with the position $\mathbf{r}_* \in \mathbb{R}^d$. But how one can define a position for a general wavepacket? Note that not every wavepacket is a particle-like one. For example, let, as before, the function $h(\mathbf{r})$ be a scalar Schwarz function, and let us consider a slightly more general than (1.3) function

$$\mathbf{h}(\beta; \mathbf{r}) = [h(\beta(\mathbf{r} - \mathbf{r}_{*1})) + h(\beta(\mathbf{r} - \mathbf{r}_{*2}))] e^{i\mathbf{k}_*\mathbf{r}} \mathbf{g}_n(\mathbf{k}_*), \quad \mathbf{r} \in \mathbb{R}^d, \quad (1.4)$$

where \mathbf{r}_{*1} and \mathbf{r}_{*2} are two arbitrary, independent vector variables. The wave $\mathbf{h}(\beta, \mathbf{r})$ defined by (1.4) is a wavepacket with the wave number \mathbf{k}_* for any choice of vectors \mathbf{r}_{*1} and \mathbf{r}_{*2} , but it is not a particle-like wavepacket since it does not have a single position \mathbf{r}_* , but rather it is a sum of two particle-like wavepackets with two positions \mathbf{r}_{*1} and \mathbf{r}_{*2} .

Our way to introduce a general particle-like wavepacket $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_{*0}; \mathbf{r})$ with a position \mathbf{r}_{*0} is by treating it as a single element of a family of wavepackets $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_*; \mathbf{r})$ with $\mathbf{r}_* \in \mathbb{R}^d$ being another *independent parameter*. In fact, we define the entire family of wavepackets $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_*; \mathbf{r})$, $\mathbf{r}_* \in \mathbb{R}^d$, subject to certain conditions allowing us to interpret any fixed $\mathbf{r}_* \in \mathbb{R}^d$ as the position of $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_*; \mathbf{r})$. Since we would like of course a wavepacket to maintain under the nonlinear evolution its particle-like property, it is clear that its definition must be sufficiently flexible to accommodate the wavepacket evolutionary variations. In light of the above discussion, the definition of the particle-like wavepacket with a transparent interpretation of its particle properties turns into the key element of the entire construction. It turns out that there is a precise description of a

particle-like wavepacket, which is rather simple and physically transparent and such a description is provided in Definition 2.2 below, see also Remarks 2.4, 2.5. The concept of the position is applicable to very general functions, it does not require a parametrization of the whole family of solutions, which was used, for example, in [25, 20, 21].

As in our previous works, we are interested in nonlinear evolution not only a single particle-like wavepacket $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_*; \mathbf{r})$, but a system $\{\mathbf{h}(\beta, \mathbf{k}_{*l}, \mathbf{r}_{*l}; \mathbf{r})\}$ of particle-like wavepackets which we call *multi-particle wavepacket*. Under certain natural conditions of genericity on \mathbf{k}_{*l} , we prove here that under the nonlinear evolution: (i) the multi-particle wavepacket remains to be a multi-particle wavepacket; (ii) the principal wavevectors \mathbf{k}_{*l} remain constant; (ii) the spatial position \mathbf{r}_{*l} of the corresponding wavepacket evolves with the constant velocity which is exactly its group velocity $\frac{1}{\varrho} \nabla \omega_n(\mathbf{k}_{*l})$. The evolution of positions of wavepackets becomes the most simple in the case, where at $\tau = 0$ we have $\mathbf{r}_{*l} = \frac{1}{\varrho} \mathbf{r}_*^0$, i.e., the case, where spatial positions are bounded in the same spatial scale in which their group velocities are bounded. In this case, the evolution of the positions is described by the formula

$$\mathbf{r}_l(\tau) = \frac{1}{\varrho} [\mathbf{r}_*^0 + \tau \nabla \omega_{n_l}(\mathbf{k}_{*l})], \quad \tau \geq 0. \quad (1.5)$$

The rectilinear motion of positions of particle-like wavepackets is a direct consequence of the spatial homogeneity of the master system (1.1). If the system were not spatially homogeneous, the motion of the positions of particle-like wavepackets would not be uniform, but we do not study that problem in this paper. In the rescaled coordinates $\mathbf{y} = \varrho \mathbf{r}$, the trajectory of every particle is a fixed, uniquely defined straight line defined uniquely if $\varrho/\beta \rightarrow 0$ as $\varrho, \beta \rightarrow 0$. Notice that under the above-mentioned genericity condition, the uniform and independent motion (1.5) of the positions of all involved particle-like wavepackets $\{\mathbf{h}(\beta, \mathbf{k}_{*l}, \mathbf{r}_{*l}; \mathbf{r})\}$ persists though they can collide in the space. In the latter case, they simply pass through each other without significant nonlinear interactions, and the nonlinear evolution with high accuracy is reduced just to a nonlinear evolution of shapes of the particle-like wavepackets. In the case, where the set of the principal wavevectors $\{\mathbf{k}_{*l}\}$ satisfy certain resonance conditions, some components of the original multi-particle wavepacket can evolve into a more complex structure which can be only partly localized in the space and, for instance, can be needle- or pancake-like. We do not study in detail those more complex structures here.

Now let us discuss in more detail the superposition principle introduced and studied for general multi-wavepackets in [8] in the particular case, where initially all $\mathbf{r}_{*l} = 0$. Here we consider multi-particle wavepackets with arbitrary \mathbf{r}_{*l} and develop a new argument based on the analysis of an averaged wavepacket interaction system introduced in [7]. Assume that the initial data \mathbf{h} for the evolution equation (1.1) is the sum of a finite number of *wavepackets (particle-like wavepackets)* \mathbf{h}_l , $l = 1, \dots, N$, i.e.,

$$\mathbf{h} = \mathbf{h}_1 + \dots + \mathbf{h}_N, \quad (1.6)$$

where the monochromaticity of every wavepacket \mathbf{h}_l is characterized by another small parameter β . The well-known *superposition principle* is a fundamental property of every linear evolutionary system, stating that the solution \mathbf{U} corresponding to the initial data \mathbf{h} as in (1.6) equals

$$\mathbf{U} = \mathbf{U}_1 + \dots + \mathbf{U}_N \text{ for } \mathbf{h} = \mathbf{h}_1 + \dots + \mathbf{h}_N, \quad (1.7)$$

where \mathbf{U}_l is the solution to the same linear problem with the initial data \mathbf{h}_l .

Evidently, the standard superposition principle cannot hold exactly as a general principle for a nonlinear system, and, at the first glance, there is no expectation for it to hold even approximately. We show though that, in fact, the superposition principle does hold with high accuracy for general dispersion nonlinear wave systems such as (1.1) provided that the initial data are a sum of generic particle-like wavepackets, and this constitutes one of the subjects of this paper. Namely, the superposition principle for nonlinear wave systems states that the solution \mathbf{U} corresponding to the multi-particle wavepacket initial data \mathbf{h} as in (1.6) satisfies

$$\mathbf{U} = \mathbf{U}_1 + \dots + \mathbf{U}_N + \mathbf{D} \text{ for } \mathbf{h} = \mathbf{h}_1 + \dots + \mathbf{h}_N, \text{ where } \mathbf{D} \text{ is small.}$$

A more detailed statement of the superposition principle for nonlinear evolution of wavepackets is as follows. We study the nonlinear evolution equation (1.1) on a finite time interval

$$0 \leq \tau \leq \tau_*, \text{ where } \tau_* > 0 \text{ is a fixed number} \quad (1.8)$$

which may depend on the L^∞ norm of the initial data \mathbf{h} but, importantly, τ_* does not depend on ϱ . We consider classes of initial data such that wave evolution governed by (1.1) is significantly nonlinear on time interval $[0, \tau_*]$ and the effect of the nonlinearity $F(\mathbf{U})$ does not vanish as $\varrho \rightarrow 0$. We assume that β, ϱ satisfy

$$0 < \beta \leq 1, \quad 0 < \varrho \leq 1, \quad \beta^2/\varrho \leq C_1 \text{ with some } C_1 > 0. \quad (1.9)$$

The above condition of boundedness on the dispersion parameter β^2/ϱ ensures that the dispersion effects are not dominant and they do not suppress nonlinear effects, see [7, 8] for a discussion.

Let us introduce the solution operator $\mathcal{S}(\mathbf{h})(\tau) : \mathbf{h} \rightarrow \mathbf{U}(\tau)$ relating the initial data \mathbf{h} of the nonlinear evolution equation (1.1) to its solution $\mathbf{U}(t)$. Suppose that the initial state is a system of particle-like wavepackets or *multi-particle wavepacket*, namely $\mathbf{h} = \sum \mathbf{h}_l$ with \mathbf{h}_l , $l = 1, \dots, N$, being “generic” wavepackets. Then for all times $0 \leq \tau \leq \tau_*$ the following *superposition principle* holds:

$$\mathcal{S}\left(\sum_{l=1}^N \mathbf{h}_l\right)(\tau) = \sum_{l=1}^N \mathcal{S}(\mathbf{h}_l)(\tau) + \mathbf{D}(\tau), \quad (1.10)$$

$$\|\mathbf{D}(\tau)\|_E = \sup_{0 \leq \tau \leq \tau_*} \|\mathbf{D}(\tau)\|_{L^\infty} \leq C_\delta \frac{\varrho}{\beta^{1+\delta}} \text{ for any small } \delta > 0. \quad (1.11)$$

Obviously, the right-hand side of (1.11) may be small only if $\varrho \leq C_1\beta$. There are examples (see [7]) in which $\mathbf{D}(\tau)$ is not small for $\varrho = C_1\beta$. In what follows, we refer to a linear combination of particle-like wavepackets as a *multi-particle wavepacket*, and to single particle-like wavepackets which constitutes the multi-particle wavepacket as *component particle wavepackets*.

Very often in theoretical studies of equations of the form (1.1) or ones reducible to it, a functional dependence between ϱ and β is imposed, resulting in a single small parameter. The most common scaling is $\varrho = \beta^2$. The nonlinear evolution of wavepackets for a variety of equations which can be reduced to the form (1.1) was studied in numerous physical and mathematical papers, mostly by asymptotic expansions of solutions with respect to a single small parameter similar to β , see [10, 12, 16, 18, 22, 24, 26, 32, 34, 36, 37] and references therein. Often the asymptotic expansions are based on a specific ansatz prescribing a certain form to the solution. In our studies here we do not use asymptotic expansions with respect to a small parameter and do not prescribe a specific form to the solution, but we impose conditions on the initial data requiring it to be a wavepacket or a linear combination of wavepackets. Since we want to establish a general property of a wide class of systems, we apply a general enough dynamical approach. There is a number of general approaches developed for the studies of high-dimensional and infinite-dimensional nonlinear evolutionary systems of hyperbolic type, see [9, 11, 17, 19, 23, 29, 33, 36, 38, 40, 41] and references therein. The approach we develop here is based on the introduction of a wavepacket interaction system. We show in [8] and here that

solutions to this system are in a close relation to solutions of the original system.

The superposition principle implies, in particular, that in the process of nonlinear evolution every single wavepacket propagates almost independently of other wavepackets (even though they may “collide” in physical space for a certain period of time) and the exact solution equals the sum of particular single wavepacket solutions with high precision. In particular, the dynamics of a solution with multi-wavepacket initial data is reduced to dynamics of separate solutions with single wavepacket data. Note that the nonlinear evolution of a single wavepacket solution for many problems is studied in detail, namely it is well approximated by its own nonlinear Schrödinger equation (NLS), see [16, 22, 26, 27, 36, 37, 38, 7] and references therein.

Let us give now an elementary physical argument justifying the superposition principle which goes as follows. If there would be no nonlinearity, the system would be linear and, consequently, the superposition principle would hold exactly. Hence any deviation from it is due to the nonlinear interactions between wavepackets, and one has to estimate their impact. Suppose that initially at time $\tau = 0$ the spatial extension s of every involved wavepacket is characterized by the parameter β^{-1} as in (1.3). Assume also (and it is quite an assumption) that the involved wavepackets evolving nonlinearly maintain somehow their wavepacket identities, including the group velocities and the spatial extensions. Then, consequently, the spatial extension of every involved wavepacket is proportional to β^{-1} and its group velocity v_l is proportional to ϱ^{-1} . The difference Δv between any two different group velocities is also proportional to ϱ^{-1} . Then the time when two different wavepackets overlap in the space is proportional to $s/|\Delta v|$ and hence to ϱ/β . Since the nonlinear term is of order one, the magnitude of the impact of the nonlinearity during this time interval should be roughly proportional to ϱ/β , which results in the same order of the magnitude of \mathbf{D} in (1.10)–(1.11). Observe that this estimate is in agreement with our rigorous estimate of the magnitude of \mathbf{D} in (1.11) if we set there $\delta = 0$.

The rigorous proof of the superposition principle presented here is not directly based on the above argument since it already implicitly relies on the principle. Though some components of the physical argument can be found in our rigorous proof. For example, we prove that the involved wavepackets maintain under the nonlinear evolution constant values of their wavevectors with well defined group velocities (the wavepacket preservation). Theorem 6.12 allows us to estimate spatial extensions of particle-like wavepackets

under the nonlinear evolution. The proof of the superposition principle for general wavepackets provided in [8] is based on general algebraic-functional considerations and on the theory of analytic operator expansions in Banach spaces. Here we develop an alternative approach with a proof based on properties of the wavepacket interaction systems introduced in [7].

To provide a flexibility in formulating more specific statements related to the spatial localization of wavepackets, we introduce a few types of wavepackets:

- a single *particle-like wavepacket* w which is characterized by the following properties: (a) its modal decomposition involves only wavevectors from β -vicinity of a single wavevector \mathbf{k}_* , where $\beta > 0$ is a small parameter; (b) it is spatially localized in all directions and can be assigned its position \mathbf{r}_* ;
- a *multi-particle wavepacket* which is a system $\{w_l\}$ of particle-like wavepackets with the corresponding sets of wavevectors $\{\mathbf{k}_{*l}\}$ and positions $\{\mathbf{r}_{*l}\}$;
- a *spatially localized multi-wavepacket* which is a system $\{w_l\}$ with w_l being either a particle-like wavepacket or a general wavepacket.

We would like to note that a more detailed analysis, which is left for another paper, indicates that, under certain resonance conditions, nonlinear interactions of particle-like wavepackets may produce a *spatially localized wavepacket* w characterized by the following properties: (i) its modal decomposition involves only wavevectors from a β -vicinity of a single wavevector \mathbf{k}_* , where $\beta > 0$ is a small parameter; (ii) it is only partly spatially localized in some, not necessarily all directions, and, for instance, it can be needle- or pancake-like.

We also would like to point out that the particular form (1.1) of the dependence on the small parameter ϱ is chosen so that appreciable nonlinear effects occur at times of order one. In fact, many important classes of problems involving small parameters can be readily reduced to the framework of (1.1) by a simple rescaling. It can be seen from the following examples. The first example is a system with a *small nonlinearity*

$$\partial_t \mathbf{v} = -i\mathbf{L}\mathbf{v} + \alpha \mathbf{f}(\mathbf{v}), \quad \mathbf{v}|_{t=0} = \mathbf{h}, \quad 0 < \alpha \ll 1, \quad (1.12)$$

where the initial data is bounded uniformly in α . Such problems are reduced to (1.1) by the time rescaling $\tau = t\alpha$. Note that here $\varrho = \alpha$ and the finite time interval $0 \leq \tau \leq \tau_*$ corresponds to the *long time interval* $0 \leq t \leq \tau_*/\alpha$.

The second example is a system with *small initial data* considered on long time intervals. The system itself has no small parameters, but the initial data are small, namely

$$\begin{aligned} \partial_t \mathbf{v} &= -i\mathbf{L}\mathbf{v} + \mathbf{f}_0(\mathbf{v}), \quad \mathbf{v}|_{t=0} = \alpha_0 \mathbf{h}, \quad 0 < \alpha_0 \ll 1, \quad \text{where} \quad (1.13) \\ \mathbf{f}_0(\mathbf{v}) &= \mathbf{f}_0^{(m)}(\mathbf{v}) + \mathbf{f}_0^{(m+1)}(\mathbf{v}) + \dots, \end{aligned}$$

where α_0 is a small parameter and $\mathbf{f}^{(m)}(\mathbf{v})$ is a homogeneous polynomial of degree $m \geq 2$. After rescaling $\mathbf{v} = \alpha_0 \mathbf{V}$ we obtain the following equation with a small nonlinearity:

$$\partial_t \mathbf{V} = -i\mathbf{L}\mathbf{V} + \alpha_0^{m-1} [\mathbf{f}_0^{(m)}(\mathbf{V}) + \alpha_0 \mathbf{f}_0^{(m+1)}(\mathbf{V}) + \dots], \quad \mathbf{V}|_{t=0} = \mathbf{h}, \quad (1.14)$$

which is of the form (1.12) with $\alpha = \alpha_0^{m-1}$. Introducing the slow time variable $\tau = t\alpha_0^{m-1}$, we get from the above an equation of the form (1.1), namely

$$\partial_\tau \mathbf{V} = -\frac{i}{\alpha_0^{m-1}} \mathbf{L}\mathbf{V} + [\mathbf{f}^{(m)}(\mathbf{V}) + \alpha_0 \mathbf{f}^{(m+1)}(\mathbf{V}) + \dots], \quad \mathbf{V}|_{t=0} = \mathbf{h}, \quad (1.15)$$

where the nonlinearity does not vanish as $\alpha_0 \rightarrow 0$. In this case, $\varrho = \alpha_0^{m-1}$ and the finite time interval $0 \leq \tau \leq \tau_*$ corresponds to the long time interval $0 \leq t \leq \tau_*/\alpha_0^{m-1}$ with small $\alpha_0 \ll 1$.

The third example is related to a *high-frequency carrier wave* in the initial data. To be concrete, we consider the nonlinear Schrödinger equation

$$\begin{aligned} \partial_\tau U - i\partial_x^2 U + i\alpha|U|^2 U, \\ U|_{\tau=0} = h_1(M\beta x)e^{iMk_{*1}x} + h_2(M\beta x)e^{iMk_{*2}x} + c.c., \end{aligned} \quad (1.16)$$

where *c.c.* stands for the complex conjugate of the prior term and $M \gg 1$ is a large parameter. Equation (1.16) can be readily recast into the form (1.1) by the change of variables $y = Mr$ yielding

$$\begin{aligned} \partial_\tau U &= -i\frac{1}{\varrho}\partial_r^2 U + i\alpha|U|^2 U, \\ U|_{\tau=0} &= h_1(\beta r)e^{ik_{*1}r} + h_2(\beta r)e^{ik_{*2}r} + c.c., \quad (1.17) \\ \text{where } \varrho &= \frac{1}{M^2} \ll 1. \end{aligned}$$

Summarizing the above analysis, we list below important ingredients of our approach.

- The wave nonlinear evolution is analyzed based on the modal decomposition with respect to the linear part of the system. The significance of the modal decomposition to the nonlinear analysis is based on the

following properties: (i) the wave modal amplitudes do not evolve under the linear evolution; (ii) the same amplitudes evolve slowly under the nonlinear evolution; (iii) modal decomposition is instrumental to the wavepacket definition including its spatial extension and the group velocity.

- Components of multi-particle wavepacket are characterized by their wavevectors \mathbf{k}_{*l} , band numbers n_l , and spatial positions \mathbf{r}_{*l} . The nonlinear evolution preserves \mathbf{k}_{*l} and n_l , whereas the spatial positions evolve uniformly with the velocities $\frac{1}{\varrho} \nabla \omega_{n_l}(\mathbf{k}_{*l})$.
- The problem involves two small parameters β and ϱ respectively in the initial data and coefficients of the master equation (1.1). These parameters scale respectively (i) the range of wavevectors involved in its modal composition, with β^{-1} scaling its spatial extension, and (ii) ϱ scaling the ratio of the slow and the fast time scales. We make no assumption on the functional dependence between β and ϱ , which are essentially independent and are subject only to inequalities.
- The nonlinear *evolution is studied for a finite time* τ_* which may depend on, say, the amplitude of the initial excitation, and, importantly, τ_* is long enough to observe appreciable nonlinear phenomena which are not vanishingly small. The superposition principle can be extended to longer time intervals up to blow-up time or even infinity if relevant uniform in β and ϱ estimates of solutions in appropriate norms are available.
- In the chosen slow time scale there are two fast wave processes with typical time scale of order ϱ which can be attributed to the linear operator \mathbf{L} : (i) fast time oscillations resulting in time averaging and consequent suppression of many nonlinear interactions; (ii) fast wavepacket propagation with large group velocities resulting in effective weakening of nonlinear interactions which are not time-averaged because of resonances. It is these two processes provide mechanisms leading to the superposition principle.

The rest of the paper is organized as follows. In the following Subsection 2.1, we introduce definitions of wavepackets, multi-wavepackets, and particle wavepackets. In Subsection 2.1, we also formulate and briefly discuss some important results of [7] which are used in this paper, and, in Subsection 2.2, we formulate new results. In Section 3, we formulate conditions imposed on the linear and the nonlinear parts of the evolution equation (1.1)

and also introduce relevant concepts describing resonance interactions inside wavepackets. In Section 4, we introduce an integral form of the basic evolution equation and study basic properties of involved operators. In Section 5, we introduce a wavepacket interaction system describing the dynamics of wavepackets. In Section 6, we first define an averaged wavepacket interaction system which plays a fundamental role in the analysis of the dynamics of multi-wavepackets and then prove that solutions to this system approximate solutions to the original equation with high accuracy. We also discuss there properties of averaged nonlinearities, in particular, for universally and conditionally universal invariant wavepackets, and prove the fundamental theorems on preservation of multi-particle wavepackets, namely Theorems 6.13 and 2.10. In Section 7, we prove the superposition principle using an approximate decoupling of the averaged wavepacket interaction system. In the last subsection of Section 7, we prove some generalizations to the cases involving nongeneric resonance interactions such as the second harmonic and third harmonic generations.

2. Statement of Results

This section consists of two subsections. In the first one, we introduce basic concepts and terminology and formulate relevant results from [7] which are used latter on, and in the second one, we formulate new results of this paper.

2.1. Wavepackets and their basic properties.

Since both linear operator $\mathbf{L}(-i\nabla)$ and nonlinearity $\mathbf{F}(\mathbf{U})$ are translation invariant, it is natural and convenient to recast the evolution equation (1.1) by applying to it the Fourier transform with respect to the space variables \mathbf{r} , namely

$$\partial_\tau \hat{\mathbf{U}}(\mathbf{k}) = -\frac{i}{\rho} \mathbf{L}(\mathbf{k}) \hat{\mathbf{U}}(\mathbf{k}) + \hat{F}(\hat{\mathbf{U}})(\mathbf{k}), \quad \hat{\mathbf{U}}(\mathbf{k}) \Big|_{\tau=0} = \hat{\mathbf{h}}(\mathbf{k}), \quad (2.1)$$

where $\hat{\mathbf{U}}(\mathbf{k})$ is the Fourier transform of $\mathbf{U}(\mathbf{r})$, i.e.,

$$\begin{aligned} \hat{\mathbf{U}}(\mathbf{k}) &= \int_{\mathbb{R}^d} \mathbf{U}(\mathbf{r}) e^{-i\mathbf{r}\cdot\mathbf{k}} d\mathbf{r}, \quad \mathbf{U}(\mathbf{r}) \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\mathbf{U}}(\mathbf{k}) e^{i\mathbf{r}\cdot\mathbf{k}} d\mathbf{k}, \quad \text{where } \mathbf{r}, \mathbf{k} \in \mathbb{R}^d, \end{aligned} \quad (2.2)$$

and \hat{F} is the Fourier form of the nonlinear operator $\mathbf{F}(\mathbf{U})$ involving convolutions, see (3.9) for details. Equation (2.1) is written in terms of the Fourier modes, and we call it the modal form of the original equation (1.1). The most of our studies are conducted first for the modal form (2.1) of the evolution equation and carried over then to the original equation (1.1).

The nonlinear evolution equations (1.1), (2.1) are commonly interpreted as describing wave propagation in a nonlinear medium. We assume that the linear part $\mathbf{L}(\mathbf{k})$ is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(\mathbf{k})$ and eigenvectors $\mathbf{g}_{n,\zeta}(\mathbf{k})$ satisfying

$$\begin{aligned} \mathbf{L}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}) &= \omega_{n,\zeta}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}), \quad \zeta = \pm, \\ \omega_{n,+}(\mathbf{k}) &\geq 0, \quad \omega_{n,-}(\mathbf{k}) \leq 0, \quad n = 1, \dots, J, \end{aligned} \quad (2.3)$$

where $\omega_{n,\zeta}(\mathbf{k})$ are real-valued, continuous for all nonsingular \mathbf{k} functions and vectors $\mathbf{g}_{n,\zeta}(\mathbf{k}) \in \mathbb{C}^{2J}$ have unit length in the standard Euclidean norm. The functions $\omega_{n,\zeta}(\mathbf{k})$, $n = 1, \dots, J$, are called *dispersion relations* between the frequency ω and the *wavevector* \mathbf{k} with n being the *band number*. We assume that the eigenvalues are naturally ordered by

$$\omega_{J,+}(\mathbf{k}) \geq \dots \geq \omega_{1,+}(\mathbf{k}) \geq 0 \geq \omega_{1,-}(\mathbf{k}) \geq \dots \geq \omega_{J,-}(\mathbf{k}) \quad (2.4)$$

and for *almost every* \mathbf{k} (with respect to the standard Lebesgue measure) the eigenvalues are distinct and, consequently, the above inequalities become strict. Importantly, we also assume the following *diagonal symmetry* condition:

$$\omega_{n,-\zeta}(-\mathbf{k}) = -\omega_{n,\zeta}(\mathbf{k}), \quad \zeta = \pm, \quad n = 1, \dots, J, \quad (2.5)$$

which is naturally presented in many physical problems (see also Remark 3.3 below) and is a fundamental condition imposed on the matrix $\mathbf{L}(\mathbf{k})$. Very often we use the abbreviation

$$\omega_{n,+}(\mathbf{k}) = \omega_n(\mathbf{k}). \quad (2.6)$$

In particular, we obtain from (2.5)

$$\omega_{n,-}(\mathbf{k}) = -\omega_n(-\mathbf{k}), \quad \omega_{n,\zeta}(\mathbf{k}) = \zeta\omega_n(\zeta\mathbf{k}), \quad \zeta = \pm. \quad (2.7)$$

In addition to that, in many examples we also have

$$\mathbf{g}_{n,\zeta}(\mathbf{k}) = \mathbf{g}_{n,-\zeta}^*(-\mathbf{k}), \quad \text{where } z^* \text{ is complex conjugate to } z. \quad (2.8)$$

We also use rather often the orthogonal projection $\Pi_{n,\zeta}(\mathbf{k})$ in \mathbb{C}^{2J} onto the complex line defined by the eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$, namely

$$\Pi_{n,\zeta}(\mathbf{k})\hat{\mathbf{u}}(\mathbf{k}) = \tilde{u}_{n,\zeta}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}), \quad n = 1, \dots, J, \quad \zeta = \pm. \quad (2.9)$$

As it is indicated by the title of this paper, we study the nonlinear problem (1.1) for initial data $\hat{\mathbf{h}}$ in the form of a properly defined particle-like *wavepackets* or, more generally, a sum of such wavepackets to which we refer as *multi-particle wavepacket*. The simplest example of a wavepacket \mathbf{w} is provided by the following formula:

$$\mathbf{w}(\beta; \mathbf{r}) = \Phi_+(\beta(\mathbf{r} - \mathbf{r}_*))e^{i\mathbf{k}_*(\mathbf{r}-\mathbf{r}_*)}\mathbf{g}_{n,+}(\mathbf{k}_*), \quad \mathbf{r} \in \mathbb{R}^d, \quad (2.10)$$

where $\mathbf{k}_* \in \mathbb{R}^d$ is a *wavepacket principal wavevector*, n is a *band number*, and $\beta > 0$ is a small parameter. We refer to the pair (n, \mathbf{k}_*) in (2.10) as a *wavepacket $n\mathbf{k}$ -pair* and \mathbf{r}_* as a *wavepacket position*. Observe that the space extension of the wavepacket $\mathbf{w}(\beta; \mathbf{r})$ is proportional to β^{-1} and it is large for small β . Notice also that, as $\beta \rightarrow 0$, the wavepacket $\mathbf{w}(\beta; \mathbf{r})$ as in (2.10) tends, up to a constant factor, to the elementary eigenmode $e^{i\mathbf{k}_*\mathbf{r}}\mathbf{g}_{n,\zeta}(\mathbf{k}_*)$ of the operator $\mathbf{L}(-i\nabla)$ with the corresponding eigenvalue $\omega_{n,\zeta}(\mathbf{k}_*)$. We refer to wavepackets of the simple form (2.10) as *simple wavepackets* to underline the very special way the parameter β enters its representation. The function $\Phi_\zeta(\mathbf{r})$, which we call *wavepacket envelope*, describes its shape, and it can be any scalar complex-valued regular enough function, for example, a function from Schwarz space. Importantly, as $\beta \rightarrow 0$, the L^∞ norm of a wavepacket (2.10) remains constant. Hence nonlinear effects in (1.1) remain strong.

Evolution of wavepackets in problems which can be reduced to the form (1.1) was studied for a variety of equations in numerous physical and mathematical papers, mostly by asymptotic expansions with respect to a single small parameter similar to β , see [10, 12, 16, 18, 22, 24, 26, 32, 34, 36, 37] and references therein. We are interested in general properties of evolutionary systems of the form (1.1) with wavepacket initial data which hold for a wide class of nonlinearities and all values of the space dimensions d and the number $2J$ of the system components. Our approach is not based on asymptotic expansions, but involves two small parameters β and ϱ with mild constraints (1.9) on their relative smallness. The constraints can be expressed in the form of either certain inequalities or equalities, and a possible simple form of such a constraint can be the power law

$$\beta = C\varrho^\varkappa, \quad \text{where } C > 0 \text{ and } \varkappa > 0 \text{ are arbitrary constants.} \quad (2.11)$$

Of course, general features of wavepacket evolution are independent of particular values of the constant C . In addition to that, some fundamental properties such as wavepacket preservation are also totally independent of the particular choice of the values of \varkappa in (2.11), whereas other properties

are independent of \varkappa as it varies in certain intervals. For instance, dispersion effects are dominant for $\varkappa < 1/2$, whereas the wavepacket superposition principle of [7] holds for $\varkappa < 1$.

To eliminate unbounded (as $\varrho \rightarrow 0$) linear term in (2.1) by replacing it with a highly oscillatory factor, we introduce the *slow variable* $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ by the formula

$$\hat{\mathbf{U}}(\mathbf{k}, \tau) = e^{-\frac{i\tau}{\varrho} \mathbf{L}(\mathbf{k})} \hat{\mathbf{u}}(\mathbf{k}, \tau) \quad (2.12)$$

and get the following equation for $\hat{\mathbf{u}}(\mathbf{k}, \tau)$:

$$\partial_\tau \hat{\mathbf{u}} = e^{\frac{i\tau}{\varrho} \mathbf{L}} \hat{\mathbf{F}}(e^{-\frac{i\tau}{\varrho} \mathbf{L}} \hat{\mathbf{u}}), \quad \hat{\mathbf{u}}|_{\tau=0} = \hat{\mathbf{h}}, \quad (2.13)$$

which, in turn, can be transformed by time integration into the integral form

$$\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}, \quad \mathcal{F}(\hat{\mathbf{u}}) = \int_0^\tau e^{\frac{i\tau'}{\varrho} \mathbf{L}} \hat{\mathbf{F}}(e^{-\frac{i\tau'}{\varrho} \mathbf{L}} \hat{\mathbf{u}}(\tau')) d\tau' \quad (2.14)$$

with an explicitly defined nonlinear polynomial integral operator $\mathcal{F} = \mathcal{F}(\varrho)$. This operator is bounded uniformly with respect to ϱ in the Banach space $E = C([0, \tau_*], L^1)$. This space has functions $\hat{\mathbf{v}}(\mathbf{k}, \tau)$, $0 \leq \tau \leq \tau_*$, as elements and has the norm

$$\|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_E = \|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_{C([0, \tau_*], L^1)} = \sup_{0 \leq \tau \leq \tau_*} \int_{\mathbb{R}^d} |\hat{\mathbf{v}}(\mathbf{k}, \tau)| d\mathbf{k}, \quad (2.15)$$

where L^1 is the Lebesgue space of functions $\hat{\mathbf{v}}(\mathbf{k})$ with the standard norm

$$\|\hat{\mathbf{v}}(\cdot)\|_{L^1} = \int_{\mathbb{R}^d} |\hat{\mathbf{v}}(\mathbf{k})| d\mathbf{k}. \quad (2.16)$$

Sometimes, we use more general weighted spaces $L^{1,a}$ with the norm

$$\|\hat{\mathbf{v}}\|_{L^{1,a}} = \int_{\mathbb{R}^d} (1 + |\mathbf{k}|)^a |\hat{\mathbf{v}}(\mathbf{k})| d\mathbf{k}, \quad a \geq 0. \quad (2.17)$$

The space $C([0, \tau_*], L^{1,a})$ with the norm

$$\|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_{E_a} = \sup_{0 \leq \tau \leq \tau_*} \int_{\mathbb{R}^d} (1 + |\mathbf{k}|)^a |\hat{\mathbf{v}}(\mathbf{k}, \tau)| d\mathbf{k} \quad (2.18)$$

is denoted by E_a , and, obviously, $E_0 = E$.

A rather elementary existence and uniqueness theorem (Theorem 4.8) implies that if $\hat{\mathbf{h}} \in L^{1,a}$, then for a small and, importantly, independent of

ϱ constant $\tau_* > 0$ this equation has a unique solution

$$\hat{\mathbf{u}}(\tau) = \mathcal{G}(\mathcal{F}(\varrho), \hat{\mathbf{h}})(\tau), \quad \tau \in [0, \tau_*], \quad \hat{\mathbf{u}} \in C^1([0, \tau_*], L^{1,a}), \quad (2.19)$$

where \mathcal{G} denotes the *solution operator* for Equation (2.14). If $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ is a solution to Equation (2.14), we call the function $\mathbf{U}(\mathbf{r}, \tau)$ defined by (2.12), (2.2) an *F-solution* to Equation (1.1). We denote by \hat{L}^1 the space of functions $\mathbf{V}(\mathbf{r})$ such that their Fourier transform $\hat{\mathbf{V}}(\mathbf{k})$ belongs to L^1 and define $\|\mathbf{V}\|_{\hat{L}^1} = \|\hat{\mathbf{V}}\|_{L^1}$. Since

$$\|\mathbf{V}\|_{L^\infty} \leq (2\pi)^{-d} \|\hat{\mathbf{V}}\|_{L^1} \quad \text{and} \quad \hat{L}^1 \subset L^\infty, \quad (2.20)$$

F-solutions to (1.1) belong to $C^1([0, \tau_*], \hat{L}^1) \subset C^1([0, \tau_*], L^\infty)$.

We would like to define wavepackets in a form which explicitly allows them to be real valued. This is accomplished based on the symmetry (2.5) of the dispersion relations, which allows us to introduce a *doublet wavepacket*

$$\begin{aligned} \mathbf{w}(\beta; \mathbf{r}) &= \Phi_+(\beta(\mathbf{r} - \mathbf{r}_*)) e^{i\mathbf{k}_*(\mathbf{r} - \mathbf{r}_*)} \mathbf{g}_{n,+}(\mathbf{k}_*) \\ &\quad + \Phi_-(\beta(\mathbf{r} - \mathbf{r}_*)) e^{-i\mathbf{k}_*(\mathbf{r} - \mathbf{r}_*)} \mathbf{g}_{n,-}(-\mathbf{k}_*). \end{aligned} \quad (2.21)$$

Such a wavepacket is real if $\Phi_-(\mathbf{r})$, $\mathbf{g}_{n,-}(-\mathbf{k}_*)$ are complex conjugate respectively to $\Phi_+(\mathbf{r})$, $\mathbf{g}_{n,+}(\mathbf{k}_*)$, i.e., if

$$\Phi_-(\mathbf{r}) = \Phi_+^*(\mathbf{r}), \quad \mathbf{g}_{n,+}(\mathbf{k}_*) = \mathbf{g}_{n,-}(-\mathbf{k}_*)^*. \quad (2.22)$$

Usually, considering wavepackets with *nk*-pair (n, \mathbf{k}_*) , we mean doublet ones as in (2.21), but sometimes we use the term wavepacket also for an elementary one as defined by (2.10). Note that the latter use is consistent with the former one since it is possible to take one of two terms in (2.21) to be zero.

Below we give a precise definition of a wavepacket. To identify characteristic properties of a wavepacket suitable for our needs, let us look at the Fourier transform $\hat{\mathbf{w}}(\beta; \mathbf{k})$ of an elementary wavepacket $\mathbf{w}(\beta; \mathbf{r})$ defined by (2.10), i.e.,

$$\hat{\mathbf{w}}(\beta; \mathbf{k}) = \beta^{-d} e^{-i\mathbf{k} \cdot \mathbf{r}_*} \hat{\Phi}(\beta^{-1}(\mathbf{k} - \mathbf{k}_*)) \mathbf{g}_{n,\zeta}(\mathbf{k}_*). \quad (2.23)$$

We call such $\hat{\mathbf{w}}(\beta; \mathbf{k})$ a wavepacket too, and assume that it possesses the following properties: (i) its L^1 norm is bounded (in fact, constant) uniformly in $\beta \rightarrow 0$; (ii) for every $\varepsilon > 0$ the value $\hat{\mathbf{w}}(\beta; \mathbf{k}) \rightarrow 0$ for every \mathbf{k} outside a $\beta^{1-\varepsilon}$ -neighborhood of \mathbf{k}_* , and the convergence is faster than any power of β if Φ is a Schwarz function. To explicitly interpret the last property, we introduce a *cutoff function* $\Psi(\eta)$ which is infinitely smooth and such that

$$\Psi(\eta) \geq 0, \quad \Psi(\eta) = 1 \text{ for } |\eta| \leq 1/2, \quad \Psi(\eta) = 0 \text{ for } |\eta| \geq 1, \quad (2.24)$$

and its shifted/rescaled modification

$$\Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \mathbf{k}_*)). \quad (2.25)$$

If an elementary wavepacket $\mathbf{w}(\beta; \mathbf{r})$ is defined by (2.23) with $\Phi(\mathbf{r})$ being a Schwarz function, then

$$\|(1 - \Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \cdot))\hat{\mathbf{w}}(\beta; \cdot)\| \leq C_{\varepsilon, s}\beta^s, \quad 0 < \beta \leq 1, \quad (2.26)$$

and the inequality holds for arbitrarily small $\varepsilon > 0$ and arbitrarily large $s > 0$. Based on the above discussion, we give the following definition of a wavepacket which is a minor variation of [8, Definiton 8].

Definition 2.1 (single-band wavepacket). Let ε be a fixed number, $0 < \varepsilon < 1$. For a given band number $n \in \{1, \dots, J\}$ and a principal wavevector $\mathbf{k}_* \in \mathbb{R}^d$ a function $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is called a *wavepacket with n -pair* (n, \mathbf{k}_*) and the degree of regularity $s > 0$ if for small $\beta < \beta_0$ with some $\beta_0 > 0$ it satisfies the following conditions: (i) $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is L^1 -bounded uniformly in β , i.e.,

$$\|\hat{\mathbf{h}}(\beta; \cdot)\|_{L^1} \leq C, \quad 0 < \beta < \beta_0 \text{ for some } C > 0; \quad (2.27)$$

(ii) $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is composed essentially of two functions $\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k})$, $\zeta = \pm$, which take values in the n th band eigenspace of $\mathbf{L}(\mathbf{k})$ and are localized near $\zeta\mathbf{k}_*$, namely

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \hat{\mathbf{h}}_-(\beta; \mathbf{k}) + \hat{\mathbf{h}}_+(\beta; \mathbf{k}) + D_h, \quad 0 < \beta < \beta_0, \quad (2.28)$$

where the components $\hat{\mathbf{h}}_\pm(\beta; \mathbf{k})$ satisfy the condition

$$\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k}) = \Psi(\beta^{1-\varepsilon}/2, \zeta\mathbf{k}_*; \mathbf{k})\Pi_{n, \zeta}(\mathbf{k})\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k}), \quad \zeta = \pm, \quad (2.29)$$

where $\Psi(\cdot, \zeta\mathbf{k}_*, \beta^{1-\varepsilon})$ is defined by (2.25) and D_h is small, namely it satisfies the inequality

$$\|D_h\|_{L^1} \leq C'\beta^s, \quad 0 < \beta < \beta_0, \text{ for some } C' > 0. \quad (2.30)$$

The inverse Fourier transform $\mathbf{h}(\beta; \mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is also called a wavepacket.

Evidently, if a wavepacket has the degree of regularity s , it also has a smaller degree of regularity $s' \leq s$ with the same ε . Observe that the degree of regularity s is related to the smoothness of $\Phi_\zeta(\mathbf{r})$ as in (2.10) so that the higher is the smoothness, the higher s/ε can be taken. Namely, if $\hat{\Phi}_\zeta \in L^{1, a}$, then one can take in (2.30) any $s < a\varepsilon$ according to the following inequality:

$$\int |(1 - \Psi(\beta^\varepsilon \eta))\hat{\Phi}_\zeta(\eta)| d\eta \leq \beta^{a\varepsilon} \|\hat{\Phi}_\zeta\|_{L^{1, a}} \leq C\beta^s. \quad (2.31)$$

For example, if we define $\hat{\mathbf{h}}_\zeta$ similarly to (2.29) and (2.23) by the formula

$$\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \mathbf{k}_*))\beta^{-d}\hat{\Phi}_\zeta(\beta^{-1}(\mathbf{k} - \mathbf{k}_*))\Pi_{n,\zeta}(\mathbf{k})\mathbf{g}, \quad (2.32)$$

where $\hat{\Phi}_\zeta(\mathbf{k})$ is a scalar Schwarz function and \mathbf{g} is a vector, then, according to (2.31), the estimate (2.30) holds and $\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k})$ is a wavepacket with arbitrarily large degree of regularity s for any given ε such that $0 < \varepsilon < 1$.

Now let us define a particle-like wavepacket following to the ideas indicated in the Introduction.

Now let us define a particle-like wavepacket following to the ideas indicated in the Introduction.

Definition 2.2 (single-band particle-like wavepacket). We call a function $\hat{\mathbf{h}}(\beta; \mathbf{k}) = \hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$, $\mathbf{r}_* \in \mathbb{R}^d$, a *particle-like wavepacket with the position \mathbf{r}_* , nk -pair (n, \mathbf{k}_*)* and the degree of regularity $s > 0$ if (i) for every \mathbf{r}_* it is a wavepacket with the degree of regularity s in the sense of the above Definition 2.1 with constants C, C' independent of $\mathbf{r}_* \in \mathbb{R}^d$; (ii) $\hat{\mathbf{h}}_\zeta$ in (2.28) satisfy the inequalities

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i\mathbf{r}'_*\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k}))|d\mathbf{k} \leq C_1\beta^{-1-\varepsilon}, \quad \zeta = \pm, \quad \mathbf{r}_* \in \mathbb{R}^d, \quad (2.33)$$

where $C_1 > 0$ is an independent of β and \mathbf{r}_* constant, ε is the same as in Definition 2.1. The inverse Fourier transform $\mathbf{h}(\beta; \mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is also called a particle-like wavepacket with the position \mathbf{r}_* . We also introduce the quantity

$$a(\mathbf{r}'_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*)) = \|\nabla_{\mathbf{k}}(e^{i\mathbf{r}'_*\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k}))\|_{L^1} \quad (2.34)$$

which we refer to as the *position detection function* for the wavepacket $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$.

Note that the left-hand side of (2.33) coincides with $a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*))$.

Remark 2.3. If $\hat{\mathbf{h}}(\beta; \mathbf{k}) = \hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ is a particle-like wavepacket with a position \mathbf{r}_* , then, applying the inverse Fourier transform to $\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k})$ and $\nabla_{\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k})$ as in (2.2), we obtain a function $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ which satisfies

$$|\mathbf{r} - \mathbf{r}_*||\mathbf{h}_\zeta(\beta, \mathbf{r}_*; \mathbf{r})| \leq (2\pi)^{-d}a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta) \quad (2.35)$$

implying that $|\mathbf{h}_\zeta(\beta; \mathbf{r})| \leq a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta)|\mathbf{r} - \mathbf{r}_*|^{-1}$. This inequality is useful for large $|\mathbf{r} - \mathbf{r}_*|$, whereas for bounded $|\mathbf{r} - \mathbf{r}_*|$ (2.27) implies the simpler inequality

$$|\mathbf{h}_\zeta(\beta, \mathbf{r}_*; \mathbf{r})| \leq (2\pi)^{-d}\|\hat{\mathbf{h}}\|_{L^1} \leq C. \quad (2.36)$$

The inequalities (2.35) and (2.33) suggest that the quantity $a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*))$ can be interpreted as a size of the particle-like wavepacket $\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k})$.

Evidently a particle-like wavepacket is a wave and not a point. Hence the above definition of its position has a degree of uncertainty, allowing, for example, to replace \mathbf{r}_* by $\mathbf{r}_* + \mathbf{a}$ with a *fixed* vector \mathbf{a} (but not allowing unbounded values of \mathbf{a}). The above definition of a particle-like wavepacket position was crafted to meet the following requirements: (i) a system of particle-like wavepackets remains to be such a system under the nonlinear evolution; (ii) it is possible (in an appropriate scale) to describe the trajectories traced out by the positions of a system of particle-wavepackets.

Remark 2.4. Typical dependence of the inverse Fourier transform $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ on \mathbf{r}_* is provided by spatial shifts by \mathbf{r}_* as in (2.21), namely

$$\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r}) = \Phi(\beta(\mathbf{r} - \mathbf{r}_*))e^{i\mathbf{k}_* \cdot (\mathbf{r} - \mathbf{r}_*)} \mathbf{g}$$

with a constant \mathbf{g} . For such a function \mathbf{h} and for any $\mathbf{r}'_* \in \mathbb{R}^d$

$$\begin{aligned} a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) &= \|\nabla_{\mathbf{k}}(\beta^{-d}e^{i\mathbf{k}\mathbf{r}'_*} \hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k}))\|_{L^1} \\ &= \|\nabla_{\mathbf{k}}(\beta^{-d}e^{i\mathbf{k}\mathbf{r}'_*} e^{-i\mathbf{k}\mathbf{r}_*} \hat{\Phi}(\mathbf{k}))\|_{L^1} \|\mathbf{g}\| \\ &= \|\mathbf{g}\| \int |i(\mathbf{r}'_* - \mathbf{r}_*) \hat{\Phi}(\mathbf{k}') + \frac{1}{\beta} \nabla_{\mathbf{k}'} \hat{\Phi}(\mathbf{k}')| d\mathbf{k}'. \end{aligned}$$

Hence, taking for simplicity $\|\mathbf{g}\| = 1$, we obtain

$$\begin{aligned} |\mathbf{r}'_* - \mathbf{r}_*| \|\hat{\Phi}\|_{L^1} + \frac{1}{\beta} \|\nabla \hat{\Phi}\|_{L^1} &\geq a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) \\ &\geq \|\mathbf{r}'_* - \mathbf{r}_*\| \|\hat{\Phi}\|_{L^1} - \frac{1}{\beta} \|\nabla \hat{\Phi}\|_{L^1}. \end{aligned} \quad (2.37)$$

For small $|\mathbf{r}'_* - \mathbf{r}_*| \ll \frac{1}{\beta}$ we see that the position detection function $a(\mathbf{r}'_*, \hat{\mathbf{h}})$ is of order $O(\beta^{-1})$, which is in the agreement with (2.33). For large $|\mathbf{r}'_* - \mathbf{r}_*| \gg \frac{1}{\beta}$ the $a(\mathbf{r}'_*, \hat{\mathbf{h}})$ is approximately proportional to $|\mathbf{r}'_* - \mathbf{r}_*|$. Therefore, if we know $a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*))$ as a function of \mathbf{r}'_* , we can recover the value of \mathbf{r}_* with the accuracy of order $O(\beta^{-1-\varepsilon})$ with arbitrary small ε . Namely, let us take arbitrary small $\varepsilon > 0$ and some $C > 0$ and consider the set

$$B(\beta) = \{\mathbf{r}'_* : a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) \leq C\beta^{-1-\varepsilon}\} \subset \mathbb{R}^d, \quad (2.38)$$

which should provide an approximate location of \mathbf{r}_* . According to (2.37), \mathbf{r}_* lies in this set for small β . If \mathbf{r}'_* lies in this set, then

$$C\beta^{-1-\varepsilon} \geq a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) \geq \|\mathbf{r}'_* - \mathbf{r}_*\| \|\hat{\Phi}\|_{L_1} - \frac{1}{\beta} \|\nabla \hat{\Phi}\|_{L_1}$$

and $|\mathbf{r}'_* - \mathbf{r}_*| \leq C_1\beta^{-1-\varepsilon} + C_2\beta^{-1}$. Hence the diameter of the $B(\beta)$ is of order $O(\beta^{-1-\varepsilon})$. Observe, taking into account Remark 2.3, that the accuracy of the wavepacket location obviously cannot be better than its size $a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*)) \sim \beta^{-1}$. The above analysis suggests that the function $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ can be viewed as *pseudoshifts* of the function $\mathbf{h}(\beta, \mathbf{0}; \mathbf{r})$ by vectors $\mathbf{r}_* \in \mathbb{R}^d$ in the sense that the regular spatial shift by \mathbf{r}_* is combined with a variation of the shape of $\mathbf{h}(\beta, \mathbf{0}; \mathbf{r})$ which is limited by the fundamental condition (2.33). In other words, according Definition 2.2, as a wavepacket moves from $\mathbf{0}$ to \mathbf{r}_* by the corresponding spatial shift, it is allowed to change its shape subject to the fundamental condition (2.33). The later is instrumental for capturing nonlinear evolution of particle-like wavepackets governed by an equation of the form (1.1).

Remark 2.5. The set $B(\beta)$ defined by (2.38) gives an approximate location of the support of the function $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ not only in the special case considered in Remark 2.4, but also when $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ is a general particle-like wavepacket. One can apply with obvious modifications the above argument for $e^{i\mathbf{k}\mathbf{r}_*} \hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ in place of $\hat{\Phi}(\mathbf{k})$ using (2.33). Here we give an alternative argument based on (2.35). Notice that the condition $a(\mathbf{r}_{*0}, \hat{\mathbf{h}}(\mathbf{r}_*)) \leq C\beta^{-1-\varepsilon}$ can be obviously satisfied not only by $\mathbf{r}_{*0} = \mathbf{r}_*$. But one can show that the diameter of the set of such \mathbf{r}_{*0} is estimated by $O(\beta^{-1-\varepsilon})$. Indeed, assume that a given function $\mathbf{h}(\beta, \mathbf{r})$ does not vanish at a given point \mathbf{r}_0 , i.e., $|\mathbf{h}(\beta, \mathbf{r}_0)| \geq c_0 > 0$ for all $\beta \leq \beta_0$. The fulfillment of (2.33) for the function $\mathbf{h}(\beta, \mathbf{r})$ with two different values of \mathbf{r}_* , namely $\mathbf{r}_* = \mathbf{r}'_*$ and $\mathbf{r}_* = \mathbf{r}''_*$ implies that

$$a(\mathbf{r}'_*, \hat{\mathbf{h}}) \leq C_1\beta^{-1-\varepsilon}, \quad a(\mathbf{r}''_*, \hat{\mathbf{h}}) \leq C_2\beta^{-1-\varepsilon},$$

and, according to (2.35), for all \mathbf{r}

$$|\mathbf{r} - \mathbf{r}'_*| |\mathbf{h}(\beta, \mathbf{r})| \leq (2\pi)^{-d} C_1 \beta^{-1-\varepsilon}, \quad |\mathbf{r} - \mathbf{r}''_*| |\mathbf{h}(\beta, \mathbf{r})| \leq (2\pi)^{-d} C_2 \beta^{-1-\varepsilon}.$$

Hence

$$|\mathbf{r}_0 - \mathbf{r}'_*| \leq \frac{(2\pi)^{-d} C_1 \beta^{-1-\varepsilon}}{c_0}, \quad |\mathbf{r}_0 - \mathbf{r}''_*| \leq \frac{(2\pi)^{-d} C_2 \beta^{-1-\varepsilon}}{c_0},$$

and

$$|\mathbf{r}'_* - \mathbf{r}''_*| \leq C_3 \beta^{-1-\varepsilon}.$$

Note that if we rescale variables \mathbf{r} and \mathbf{r}_* as in Example 2.13, namely $\varrho\mathbf{r} = \mathbf{y}$ and $\varrho\mathbf{r}_* = \mathbf{y}_*$ with $\varrho = \beta^2$, the diameter of the set $B(\beta)$ in the y -coordinates is of order $\beta^{1-\varepsilon} \ll 1$, and hence this set gives a good approximation for the location of the particle-like wavepacket as $\beta \rightarrow 0$. It is important to notice that our method to locate the support of wavepackets is applicable to very general wavepackets and does not use their specific form. This flexibility allows us to prove that particle-like wavepackets and their positions are well defined during nonlinear dynamics of generic equations with rather general initial data which form infinite-dimensional function spaces. Another approaches to describe dynamics of waves are applied to situations, where solutions under considerations can be parametrized by a finite number of parameters and the dynamics of parameters describes the dynamics of the solutions. See for example [25], [20], [21], where dynamics of centers of solutions is described.

Remark 2.6. Note that for a single wavepacket initial data $\mathbf{h}(\beta, \mathbf{r} - \mathbf{r}'_*)$ one can make a change of variables to a moving frame (\mathbf{x}, τ) , namely $(\mathbf{r}, \tau) = (\mathbf{x} + \mathbf{v}\tau, \tau)$, where $\mathbf{v} = \frac{1}{\varrho}\nabla\omega(\mathbf{k}_*)$ is the group velocity; this change of variables makes the group velocity zero. Often it is possible to prove that dynamics preserves functions which decay at infinity, namely if the initial data $\mathbf{h}(\beta, \mathbf{x})$ decays at the spatial infinity, then the solution $\mathbf{U}(\beta, \mathbf{x}, \tau)$ also decays at infinity (though the corresponding proofs can be rather technical). This property can be reformulated in rescaled \mathbf{y} variables as follows: if initial data are localized about zero, then the solution is localized about zero as well. Then, using the fact that the equation has constant coefficients, we observe that the solution $\mathbf{U}(\beta, \mathbf{y} - \mathbf{y}'_*, \tau)$ corresponding to $\mathbf{h}(\beta, \mathbf{y} - \mathbf{y}'_*)$ is localized about \mathbf{y}'_* provided that $\mathbf{h}(\beta, \mathbf{y})$ was localized about the origin. Note that, in this paper, we consider the much more complicated case of multiple wavepackets. Even in the simplest case of the initial multi-wavepacket which involves only two components, namely the wavepacket $\mathbf{h}(\beta, \mathbf{r}) = \mathbf{h}_1(\beta, \mathbf{r} - \mathbf{r}'_*) + \mathbf{h}_2(\beta, \mathbf{r} - \mathbf{r}''_*)$ with two principal wave vectors $\mathbf{k}_{1*} \neq \mathbf{k}_{2*}$, it is evident that one cannot use the above considerations based on the change of variables and the translational invariance. Using other arguments developed in this paper, we prove that systems of particle-like wavepackets remain localized in the process of the nonlinear evolution.

Note that similarly to (1.2) and (1.4) a function of the form

$$\beta^{-d}(e^{-i\mathbf{k}\mathbf{r}_{*1}} + e^{-i\mathbf{k}\mathbf{r}_{*2}}) \left[\hat{h}\left(\frac{\mathbf{k} - \mathbf{k}_*}{\beta}\right) \right] \mathbf{g}_n(\mathbf{k}_*),$$

defined for any pair of \mathbf{r}_{*1} and \mathbf{r}_{*2} , where \hat{h} is a Schwarz function and all constants in Definition 2.1 are independent of $\mathbf{r}_{*1}, \mathbf{r}_{*2} \in \mathbb{R}^d$, is not a single particle-like wavepacket since it does not have a single wavepacket position \mathbf{r}_* , but rather it is a sum of two particle-like wavepackets with two positions \mathbf{r}_{*1} and \mathbf{r}_{*2} .

We want to emphasize once more that a particle-like wavepacket is defined as the family $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ with \mathbf{r}_* being an independent variable running the entire space \mathbb{R}^d , see, for example, (1.2), (1.3), and (2.21). In particular, we can choose a dependence of \mathbf{r}_* on β and ϱ . An interesting type of such a dependence is $\mathbf{r}_* = \mathbf{r}_*^0/\varrho$, where ϱ satisfies (2.11) as we discuss below in Example 2.13.

Our special interest is in the waves that are finite sums of wavepackets which we refer to as *multi-wavepackets*.

Definition 2.7 (multi-wavepacket). Let S be a set of nk -pairs:

$$S = \{(n_l, \mathbf{k}_{*l}), l = 1, \dots, N\} \subset \Sigma = \{1, \dots, J\} \times \mathbb{R}^d, \quad (2.39)$$

$$(n_l, \mathbf{k}_{*l}) \neq (n_{l'}, \mathbf{k}_{*l'}) \text{ for } l \neq l',$$

and let $N = |S|$ be their number. Let K_S be a set consisting of all different wavevectors \mathbf{k}_{*l} involved in S with $|K_S| \leq N$ being the number of its elements. K_S is called a *wavepacket k -spectrum* and, without loss of generality, we assume the indexing of elements (n_l, \mathbf{k}_{*l}) in S to be such that

$$K_S = \{\mathbf{k}_{*i}, i = 1, \dots, |K_S|\}, \text{ i.e., } l = i \text{ for } 1 \leq i \leq |K_S|. \quad (2.40)$$

A function $\hat{\mathbf{h}}(\beta) = \hat{\mathbf{h}}(\beta; \mathbf{k})$ is called a *multi-wavepacket* with *nk -spectrum* S if it is a finite sum of wavepackets, namely

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \sum_{l=1}^N \hat{\mathbf{h}}_l(\beta; \mathbf{k}), \quad 0 < \beta < \beta_0 \text{ for some } \beta_0 > 0, \quad (2.41)$$

where $\hat{\mathbf{h}}_l$, $l = 1, \dots, N$, is a wavepacket with nk -pair $(\mathbf{k}_{*l}, n_l) \in S$ as in Definition 2.1. If all the wavepackets $\hat{\mathbf{h}}_l(\beta; \mathbf{k}) = \hat{\mathbf{h}}_l(\beta, \mathbf{r}_{*l}; \mathbf{k})$ are particle-like ones with respective positions \mathbf{r}_{*l} , then the multi-wavepacket is called *multi-particle wavepacket* and we refer to $(\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N})$ as its position vector.

Note that if $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is a wavepacket, then $\hat{\mathbf{h}}(\beta; \mathbf{k}) + O(\beta^s)$ is also a wavepacket with the same nk -spectrum, and the same is true for multi-wavepackets. Hence we can introduce multi-wavepackets equivalence relation “ \simeq ” of the degree s by

$$\hat{\mathbf{h}}_1(\beta; \mathbf{k}) \simeq \hat{\mathbf{h}}_2(\beta; \mathbf{k}) \text{ if } \|\hat{\mathbf{h}}_1(\beta; \mathbf{k}) - \hat{\mathbf{h}}_2(\beta; \mathbf{k})\|_{L^1} \leq C\beta^s \quad (2.42)$$

for some constant $C > 0$.

Note that the condition (2.33) does not impose restrictions on the term D_h in (2.28). Therefore, this equivalence can be applied to particle wavepackets.

Let us turn now to the abstract nonlinear problem (2.14), where (i) $\mathcal{F} = \mathcal{F}(\varrho)$ depends on ϱ and (ii) the initial data $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta)$ is a multi-wavepacket depending on β . We would like to state our first theorem on multi-wavepacket preservation under the evolution (2.14) as $\beta, \varrho \rightarrow 0$, which holds provided its nk -spectrum S satisfies a natural condition called *resonance invariance*. This condition is intimately related to the so-called *phase* and *frequency matching* conditions for stronger nonlinear interactions, and its concise formulation is as follows. We define for given dispersion relations $\{\omega_n(\mathbf{k})\}$ and any finite set $S \subset \{1, \dots, J\} \times \mathbb{R}^d$ another finite set $\mathcal{R}(S) \subset \{1, \dots, J\} \times \mathbb{R}^d$, where \mathcal{R} is a certain algebraic operation described in Definition 3.8 below. It turns out that for any S always $S \subseteq \mathcal{R}(S)$, but if $\mathcal{R}(S) = S$ we call S *resonance invariant*. The condition of resonance invariance is instrumental for the multi-wavepacket preservation, and there are examples showing that if it fails, i.e., $\mathcal{R}(S) \neq S$, the wavepacket preservation does not hold. Importantly, the resonance invariance $\mathcal{R}(S) = S$ allows resonances inside the multi-wavepacket, that includes, in particular, resonances associated with the second and third harmonic generations, resonant four-wave interaction, etc. In this paper, we use basic results on wavepacket preservation obtained in [7], and we formulate theorems from [7] we need here. Since we use constructions from [7], for completeness we provide also their proofs in the following subsections. The following two theorems are proved in [7].

Theorem 2.8 (multi-wavepacket preservation). *Suppose that the nonlinear evolution is governed by (2.14) and the initial data $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta; \mathbf{k})$ is a multi-wavepacket with nk -spectrum S and the regularity degree s . Assume that S is resonance invariant in the sense of Definition 3.8 below. Let $\rho(\beta)$ be any function satisfying*

$$0 < \rho(\beta) \leq C\beta^s \text{ for some constant } C > 0, \quad (2.43)$$

and let us set $\varrho = \rho(\beta)$. Then the solution $\hat{\mathbf{u}}(\tau, \beta) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ to (2.14) for any $\tau \in [0, \tau_*]$ is a multi-wavepacket with nk -spectrum S and the regularity degree s , i.e.,

$$\hat{\mathbf{u}}(\tau, \beta; \mathbf{k}) = \sum_{l=1}^N \hat{\mathbf{u}}_l(\tau, \beta; \mathbf{k}), \tag{2.44}$$

where $\hat{\mathbf{u}}_l$ is a wavepacket with nk -pair $(n_l, \mathbf{k}_{*l}) \in S$.

The time interval length $\tau_* > 0$ depends only on the L^1 -norms of $\hat{\mathbf{h}}_l(\beta; \mathbf{k})$ and N . The presentation (2.44) is unique up to the equivalence (2.42) of degree s .

The above statement can be interpreted as follows. Modes in nk -spectrum S are always resonance coupled with modes in $\mathcal{R}(S)$ through the nonlinear interactions, but if $\mathcal{R}(S) = S$, then (i) all resonance interactions occur inside S and (ii) only small vicinity of S is involved in nonlinear interactions leading to the multi-wavepacket preservation.

The statement of Theorems 2.8 directly follows from the following general theorem proved in [7].

Theorem 2.9 (multi-wavepacket approximation). *Let the initial data $\hat{\mathbf{h}}$ in the integral equation (2.14) be a multi-wavepacket $\hat{\mathbf{h}}(\beta; \mathbf{k})$ with nk -spectrum S as in (2.39), regularity degree s , and parameter $\varepsilon > 0$ as in Definition 2.1. Assume that S is resonance invariant in the sense of Definition 3.8 below. Let the cutoff function $\Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \mathbf{k})$ and the eigenvector projectors $\Pi_{n,\pm}(\mathbf{k})$ be defined by (2.25) and (2.9) respectively. For a solution $\hat{\mathbf{u}}$ of (2.14) we set*

$$\hat{\mathbf{u}}_l(\beta; \tau, \mathbf{k}) = \left[\sum_{\zeta=\pm} \Psi(C\beta^{1-\varepsilon}, \zeta\mathbf{k}_{*l}; \mathbf{k}) \Pi_{n_l, \zeta}(\mathbf{k}) \right] \hat{\mathbf{u}}(\beta; \tau, \mathbf{k}), \tag{2.45}$$

$$l = 1, \dots, N.$$

Then every such $\hat{\mathbf{u}}_l(\beta; \tau, \mathbf{k})$ is a wavepacket and

$$\sup_{0 \leq \tau \leq \tau_*} \left\| \hat{\mathbf{u}}(\beta; \tau, \mathbf{k}) - \sum_{l=1}^N \hat{\mathbf{u}}_l(\beta; \tau, \mathbf{k}) \right\|_{L^1} \leq C_1 \varrho + C_2 \beta^s, \tag{2.46}$$

where the constants C, C_1 do not depend on ε, s, ρ , and β and the constant C_2 does not depend on ρ and β .

We would like to point out also that Theorem 2.8 allows us to take values $\hat{\mathbf{u}}(\tau_*)$ as new wavepacket initial data for (1.1) and extend the wavepacket

invariance of a solution to the next time interval $\tau_* \leq \tau \leq \tau_{*1}$. This observation allows us to extend the wavepacket invariance to larger values of τ (up to blow-up time or infinity) if some additional information about solutions with wavepacket initial data is available, see [7].

Note that the wavepacket form of solutions can be used to obtain long-time estimates of solutions. Namely, very often the behavior of every single wavepacket is well approximated by its own nonlinear Schrödinger equation (NLS), see [15, 30, 16, 22, 26, 27, 34, 36, 37, 38] and references therein, see also Section 6. Many features of the dynamics governed by NLS-type equations are well understood, see [13, 14, 31, 35, 39, 40] and references therein. These results can be used to obtain long-time estimates for every single wavepacket (as, for example, in [27]) and, with the help of the superposition principle, for the multi-wavepacket solution.

2.2. Formulation of new results on particle wavepackets.

In this paper, we prove the following refinement of Theorem 2.8 for the case of multi-particle wavepackets.

Theorem 2.10 (multi-particle wavepacket preservation). *Assume that the conditions of Theorem 2.9 hold and, in addition to that, the initial data $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta; \mathbf{k})$ is a multi-particle wavepacket of degree s with positions $\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N}$ and the multi-particle wavepacket is universally resonance invariant in the sense of Definition 3.8. Assume also that*

$$\rho(\beta) \leq C\beta^{s_0}, s_0 > 0. \quad (2.47)$$

Then the solution $\hat{\mathbf{u}}(\beta; \tau) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ to (2.14) for any $\tau \in [0, \tau_]$ is a multi-particle wavepacket with the same nk -spectrum S and the same positions $\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N}$. Namely, (2.46) holds, where $\hat{\mathbf{u}}_l$ is a wavepacket with nk -pair $(n_l, \mathbf{k}_{*l}) \in S$ defined by (2.45), the constants C, C_1, C_2 do not depend on \mathbf{r}_{*l} , and every $\hat{\mathbf{u}}_l$ is equivalent in the sense of the equivalence (2.42) of degree $s_1 = \min(s, s_0)$ to a particle wavepacket with the position \mathbf{r}_{*l} .*

Remark 2.11. Note that in the statement of the above theorem the positions $\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N}$ of wavepackets which compose the solution $\hat{\mathbf{u}}(\beta; \tau, \mathbf{k})$ of (2.13) and (2.14) do not depend on τ and, hence, do not move. Note also that the solution $\hat{\mathbf{U}}(\beta; \tau, \mathbf{k})$ of the original equation (2.1), related to $\hat{\mathbf{u}}(\beta; \tau, \mathbf{k})$ by the change of variables (2.12), is composed of wavepackets $\mathbf{U}_l(\beta; \tau, \mathbf{r})$ corresponding to $\mathbf{u}_l(\beta; \tau, \mathbf{r})$, have their positions moving with

respective constant velocities $\nabla_k \omega(\mathbf{k}_{*l})$ (see for details Remark 4.1, see also the following corollary).

Using Proposition 4.2, we obtain from Theorem 2.10 the following corollary.

Corollary 2.12. *Let the conditions of Theorem 2.10 hold, and let $\hat{\mathbf{U}}(\beta; \tau, \mathbf{k})$ be defined by (2.12) in terms of $\hat{\mathbf{u}}(\beta; \tau, \mathbf{k})$. Let*

$$\frac{\beta^2}{\varrho} \leq C, \text{ with some } C, \quad 0 < \beta \leq \frac{1}{2}, \quad 0 < \varrho \leq \frac{1}{2}. \quad (2.48)$$

Then $\hat{\mathbf{U}}(\beta; \tau, \mathbf{k})$ is for every $\tau \in [0, \tau_]$ a particle multi-wavepacket in the sense of Definition 2.2 with the same $n\mathbf{k}$ -spectrum S , regularity degree s_1 , and τ -dependent positions $\mathbf{r}_{*l} + \frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_{*l})$.*

In the following example, we consider the case, where spatial positions of wavepackets have a specific dependence on parameter ϱ , namely $\mathbf{r}_* = \mathbf{r}_{*l}^0 / \varrho$.

Example 2.13 (wavepacket trajectories and collisions). Let us rescale the coordinates in the physical space as follows:

$$\varrho \mathbf{r} = \mathbf{y} \quad (2.49)$$

with the consequent rescaling of the wavevector variable (dual with respect to Fourier transform) $\mathbf{k} = \varrho \boldsymbol{\eta}$. It follows then that under the evolution (1.1) the group velocity of a wavepacket with a wavevector \mathbf{k}_* in the new coordinates \mathbf{y} becomes $\nabla_k \omega(\mathbf{k}_*)$ and, evidently, is of order one. If we set the positions $\mathbf{r}_{*l} = \mathbf{r}_{*l}^0 / \varrho$ with fixed \mathbf{r}_{*l}^0 , then, according to (2.35), the wavepackets $|\mathbf{h}(\beta; \mathbf{r})|$ in \mathbf{y} -variables have characteristic spatial scale $\mathbf{y} - \mathbf{r}_{*l}^0 \sim \varrho a(\mathbf{r}_{*l}, \hat{\mathbf{h}}) \sim \varrho \beta^{-1}$ which is small if ϱ / β is small. The positions of particle-like wavepackets (quasiparticles) $\hat{\mathbf{U}}(\mathbf{y} / \varrho, \tau)$ are initially located at $\mathbf{y}_l = \mathbf{r}_{*l}^0$ and propagate with the group velocities $\nabla_k \omega(\mathbf{k}_{*l})$. Their trajectories are straight lines in the space \mathbb{R}^d described by

$$\mathbf{y} = \tau \nabla_k \omega(\mathbf{k}_{*l}) + \mathbf{r}_{*l}^0, \quad 0 \leq \tau \leq \tau_*,$$

(compare with (1.5)). The trajectories may intersect, indicating ‘‘collisions’’ of quasiparticles. Our results (Theorem 2.10) show that if a multi-particle wavepacket initially was universally resonance invariant, then the involved particle-like wavepackets preserve their identity in spite of collisions and the fact that the nonlinear interactions with other wavepackets (quasiparticles) are not small; in fact, they are of order one. Note that \mathbf{r}_{*l}^0 can be chosen

arbitrarily implying that up to $N(N - 1)$ collisions can occur on the time interval $[0, \tau_*]$ on which we study the system evolution.

To formulate the approximate superposition principle for multi-particle wavepackets, we introduce now the *solution operator* \mathcal{G} mapping the initial data $\hat{\mathbf{h}}$ into the solution $\hat{\mathbf{U}} = \mathcal{G}(\hat{\mathbf{h}})$ of the modal evolution equation (2.14). This operator is defined for $\|\hat{\mathbf{h}}\| \leq R$ according to the existence and uniqueness Theorem 4.7. The main result of this paper is the following statement.

Theorem 2.14 (superposition principle). *Suppose that the initial data $\hat{\mathbf{h}}$ of (2.14) is a multi-particle wavepacket of the form*

$$\hat{\mathbf{h}} = \sum_{l=1}^N \hat{\mathbf{h}}_l, \quad N \max_l \|\hat{\mathbf{h}}_l\|_{L^1} \leq R, \quad (2.50)$$

satisfying Definition 2.7 and its nk -spectrum is universally resonance invariant in the sense of Definition 3.8. Suppose also that the group velocities of wavepackets are different, namely

$$\nabla_{\mathbf{k}} \omega_{n_{l_1}}(\mathbf{k}_{*l_1}) \neq \nabla_{\mathbf{k}} \omega_{n_{l_2}}(\mathbf{k}_{*l_2}) \text{ if } l_1 \neq l_2 \quad (2.51)$$

and that (2.48) holds. Then the solution $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ to the evolution equation (2.14) satisfies the following approximate superposition principle:

$$\mathcal{G}\left(\sum_{l=1}^{N_h} \hat{\mathbf{h}}_l\right) = \sum_{l=1}^{N_h} \mathcal{G}(\hat{\mathbf{h}}_l) + \tilde{\mathbf{D}}, \quad (2.52)$$

with a small remainder $\tilde{\mathbf{D}}(\tau)$ such that

$$\sup_{0 \leq \tau \leq \tau_*} \|\tilde{\mathbf{D}}(\tau)\|_{L^1} \leq C_\varepsilon \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|, \quad (2.53)$$

where (i) ε is the same as in Definition 2.1 and can be arbitrary small; (ii) τ_ does not depend on β , ϱ , \mathbf{r}_{*l} , and ε ; (iii) C_ε does not depend on β , ϱ , and positions \mathbf{r}_{*l} .*

A particular case of the above theorem in which there was no dependence on \mathbf{r}_{*l} was proved in [8] by a different method based on the theory of analytic operators in Banach spaces. The condition (2.51) can be relaxed if the initial positions of involved particle-like wavepackets are far apart, and the corresponding results are formulated in the theorem below and in Example 2.13.

Theorem 2.15 (superposition principle). *Suppose that the initial data $\hat{\mathbf{h}}$ of (2.14) is a multi-particle wavepacket of the form (2.50) with a*

universally resonance invariant nk -spectrum in the sense of Definition 3.8 and (2.48) holds. Suppose also that either the group velocities of wavepackets are different, namely (2.51) holds, or the positions \mathbf{r}_{*l} satisfy the inequality

$$\tau_* |\mathbf{r}_{*l_1} - \mathbf{r}_{*l_2}|^{-1} \leq \frac{\varrho}{2C_{\omega,2}\beta^{1-\varepsilon}} \quad (2.54)$$

if $\nabla_{\mathbf{k}}\omega_{n_{l_1}}(\mathbf{k}_{*l_1}) = \nabla_{\mathbf{k}}\omega_{n_{l_2}}(\mathbf{k}_{*l_2})$, $l_1 \neq l_2$,

where the constant $C_{\omega,2}$ is the same as in (3.2). Then the solution $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ to the evolution equation (2.14) satisfies the approximate superposition principle (2.52), (2.53).

We prove in this paper further generalizations of the particle-like wavepacket preservation and the superposition principle to the cases, where the nk -spectrum of a multi-wavepacket is not universal resonance invariant such as the cases of multi-wavepackets involving the second and third harmonic generation. In particular, we prove Theorem 7.5 showing that many (but, may be, not all) components of involved wavepackets remain spatially localized. Another Theorem 7.7 extends the superposition principle to the case, where resonance interactions between components of a multi-wavepackets can occur.

3. Conditions and Definitions

In this section, we formulate and discuss all definitions and conditions under which we study the nonlinear evolutionary system (1.1) through its modal, Fourier form (2.1). Most of the conditions and definitions are naturally formulated for the modal form (2.1), and this is one of the reasons we use it as the basic one.

3.1. Linear part.

The basic properties of the linear part $\mathbf{L}(\mathbf{k})$ of the system (2.1), which is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(\mathbf{k})$, has been already discussed in the Introduction. To account for all needed properties of $\mathbf{L}(\mathbf{k})$, we define the singular set of points \mathbf{k} .

Definition 3.1 (band-crossing points). We call \mathbf{k}_0 a *band-crossing point* for $\mathbf{L}(\mathbf{k})$ if $\omega_{n+1,\zeta}(\mathbf{k}_0) = \omega_{n,\zeta}(\mathbf{k}_0)$ for some n, ζ or $\mathbf{L}(\mathbf{k})$ is not continuous at \mathbf{k}_0 or if $\omega_{1,\pm}(\mathbf{k}_0) = 0$. The set of such points is denoted by σ_{bc} .

In the next condition, we collect all constraints imposed on the linear operator $\mathbf{L}(\mathbf{k})$.

Condition 3.2 (linear part). *The linear part $\mathbf{L}(\mathbf{k})$ of the system (2.1) is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(\mathbf{k})$ and corresponding eigenvectors $\mathbf{g}_{n,\zeta}(\mathbf{k})$ satisfying for $\mathbf{k} \notin \sigma_{bc}$ the basic relations (2.3)-(2.5). In addition to that, we assume:*

- (i) *the set of band-crossing points σ_{bc} is a closed, nowhere dense set in \mathbb{R}^d and has zero Lebesgue measure;*
- (ii) *the entries of the Hermitian matrix $\mathbf{L}(\mathbf{k})$ are infinitely differentiable in \mathbf{k} for all $\mathbf{k} \notin \sigma_{bc}$ that readily implies via the spectral theory, [28], infinite differentiability of all eigenvalues $\omega_n(\mathbf{k})$ in \mathbf{k} for all $\mathbf{k} \notin \sigma_{bc}$;*
- (iii) $\mathbf{L}(\mathbf{k})$ *satisfies the polynomial bound*

$$\|\mathbf{L}(\mathbf{k})\| \leq C(1 + |\mathbf{k}|^p), \quad \mathbf{k} \in \mathbb{R}^d, \quad \text{for some } C > 0 \text{ and } p > 0. \quad (3.1)$$

Note that since $\omega_{n,\zeta}(\mathbf{k})$ are smooth if $\mathbf{k} \notin \sigma_{bc}$, the following relations hold for any (n, k) -spectrum S :

$$\begin{aligned} \max_{|\mathbf{k} \pm \mathbf{k}_{*l}| \leq \pi_0, \quad l=1, \dots, N,} |\nabla_{\mathbf{k}} \omega_{n_l, \zeta}| &\leq C_{\omega, 1}, \\ \max_{|\mathbf{k} \pm \mathbf{k}_{*l}| \leq \pi_0, \quad l=1, \dots, N,} |\nabla_{\mathbf{k}}^2 \omega_{n_l, \zeta}| &\leq C_{\omega, 2}, \end{aligned} \quad (3.2)$$

where $C_{\omega, 1}$ and $C_{\omega, 2}$ are positive constants and

$$\pi_0 = \frac{1}{2} \min_{l=1, \dots, N} \min(\text{dist}\{\pm \mathbf{k}_{*l}, \sigma_{bc}\}, 1). \quad (3.3)$$

Remark 3.3 (dispersion relations symmetry). The symmetry condition (2.5) on the dispersion relations naturally arise in many physical problems, for example, the Maxwell equations in periodic media, see [1]–[3], [5], or when $\mathbf{L}(\mathbf{k})$ originates from a Hamiltonian. We would like to stress that this symmetry conditions are not imposed to simplify studies, but rather to take into account fundamental symmetries of physical media. The symmetry causes resonant nonlinear interactions, which create non-trivial effects. Interestingly, many problems without symmetries can be put into the framework with the symmetry by a certain extension, [7].

Remark 3.4 (band-crossing points). Band-crossing points are discussed in more detail in [1, Section 5.4], [2, Sections 4.1, 4.2]. In particular, generically the set σ_{bc} of band-crossing point is a manifold of dimension $d - 2$. Notice also that there is a natural ambiguity in the definition of a normalized eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$ of $\mathbf{L}(\mathbf{k})$ which is defined up to a complex

number ξ with $|\xi| = 1$. This ambiguity may not allow an eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$ which can be a locally smooth function in \mathbf{k} to be a uniquely defined continuous function in \mathbf{k} globally for all $\mathbf{k} \notin \sigma_{bc}$ because of a possibility of branching. But, importantly, the orthogonal projector $\Pi_{n,\zeta}(\mathbf{k})$ on $\mathbf{g}_{n,\zeta}(\mathbf{k})$ as defined by (2.9) is uniquely defined and, consequently, infinitely differentiable in \mathbf{k} via the spectral theory, [28], for all $\mathbf{k} \notin \sigma_{bc}$. Since we consider $\hat{\mathbf{U}}(\mathbf{k})$ as an element of the space L^1 and σ_{bc} is of zero Lebesgue measure, considering $\mathbf{k} \notin \sigma_{bc}$ is sufficient for us.

We introduce for vectors $\hat{\mathbf{u}} \in \mathbb{C}^{2J}$ their expansion with respect to the orthonormal basis $\{\mathbf{g}_{n,\zeta}(\mathbf{k})\}$:

$$\begin{aligned} \hat{\mathbf{u}}(\mathbf{k}) &= \sum_{n=1}^J \sum_{\zeta=\pm} \hat{u}_{n,\zeta}(\mathbf{k}) \mathbf{g}_{n,\zeta}(\mathbf{k}) \\ &= \sum_{n=1}^J \sum_{\zeta=\pm} \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}), \quad \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}) = \Pi_{n,\zeta}(\mathbf{k}) \hat{\mathbf{u}}(\mathbf{k}) \end{aligned} \quad (3.4)$$

and we refer to it as the *modal decomposition* of $\hat{\mathbf{u}}(\mathbf{k})$ and to $\hat{u}_{n,\zeta}(\mathbf{k})$ as the *modal coefficients* of $\hat{\mathbf{u}}(\mathbf{k})$. Evidently,

$$\sum_{n=1}^J \sum_{\zeta=\pm} \Pi_{n,\zeta}(\mathbf{k}) = I_{2J}, \quad (3.5)$$

where I_{2J} is the $2J \times 2J$ identity matrix.

Notice that we can define the action of the operator $\mathbf{L}(-i\nabla_{\mathbf{r}})$ on any Schwarz function $\mathbf{Y}(\mathbf{r})$ by the formula

$$\mathbf{L}(\widehat{-i\nabla_{\mathbf{r}}}) \mathbf{Y}(\mathbf{k}) = \mathbf{L}(\mathbf{k}) \hat{\mathbf{Y}}(\mathbf{k}), \quad (3.6)$$

where, in view of the polynomial bound (3.1), the order of \mathbf{L} does not exceed p . In a special case, where all the entries of $\mathbf{L}(\mathbf{k})$ are polynomials, (3.6) turns into the action of the differential operator with constant coefficients.

3.2. Nonlinear part.

The nonlinear term \hat{F} in (2.1) is assumed to be a general functional polynomial of the form

$$\hat{F}(\hat{\mathbf{U}}) = \sum_{m \in \mathfrak{M}_F} \hat{F}^{(m)}(\hat{\mathbf{U}}^m), \quad (3.7)$$

where $\hat{F}^{(m)}$ is an m -homogeneous polylinear operator,

$$\begin{aligned} \mathfrak{M}_F &= \{m_1, \dots, m_p\} \subset \{2, 3, \dots\} \text{ is a finite set,} \\ &\text{and } m_F = \max\{m : m \in \mathfrak{M}_F\}. \end{aligned} \quad (3.8)$$

The integer m_F in (3.8) is called the *degree of the functional polynomial* \hat{F} . For instance, if $\mathfrak{M}_F = \{2\}$ or $\mathfrak{M}_F = \{3\}$, the polynomial \hat{F} is respectively homogeneous quadratic or cubic. Every m -linear operator $\hat{F}^{(m)}$ in (3.7) is assumed to be of the form of a convolution

$$\begin{aligned} &\hat{F}^{(m)}(\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_m)(\mathbf{k}, \tau) \\ &= \int_{\mathbb{D}_m} \chi^{(m)}(\mathbf{k}, \vec{k}) \hat{\mathbf{U}}_1(\mathbf{k}') \dots \hat{\mathbf{U}}_m(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \tilde{d}^{(m-1)d} \vec{k}, \end{aligned} \quad (3.9)$$

$$\text{where } \mathbb{D}_m = \mathbb{R}^{(m-1)d}, \quad \tilde{d}^{(m-1)d} \vec{k} = \frac{d\mathbf{k}' \dots d\mathbf{k}^{(m-1)}}{(2\pi)^{(m-1)d}},$$

$$\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}) = \mathbf{k} - \mathbf{k}' - \dots - \mathbf{k}^{(m-1)}, \quad \vec{k} = (\mathbf{k}', \dots, \mathbf{k}^{(m)}). \quad (3.10)$$

indicating that the nonlinear operator $F^{(m)}(\mathbf{U}_1, \dots, \mathbf{U}_m)$ is translation invariant (it may be local or nonlocal). The quantities $\chi^{(m)}$ in (3.9) are called *susceptibilities*. For numerous examples of nonlinearities of the form similar to (3.7), (3.9) see [1]–[7] and references therein. In what follows, the nonlinear term \hat{F} in (2.1) will satisfy the following conditions.

Condition 3.5 (nonlinearity). *The nonlinearity $\hat{F}(\hat{\mathbf{U}})$ is assumed to be of the form (3.7)–(3.9). The susceptibility $\chi^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})$ is infinitely differentiable for all \mathbf{k} and $\mathbf{k}^{(j)}$ which are not band-crossing points, and is bounded, namely for some constant C_χ*

$$\begin{aligned} \|\chi^{(m)}\| &= (2\pi)^{-(m-1)d} \sup_{\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)} \in \mathbb{R}^d \setminus \sigma_{bc}} |\chi^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})| \\ &\leq C_\chi, \quad m \in \mathfrak{M}_F, \end{aligned} \quad (3.11)$$

where the norm $|\chi^{(m)}(\mathbf{k}, \vec{k})|$ of the m -linear tensor $\chi^{(m)} : (\mathbb{C}^{2J})^m \rightarrow (\mathbb{C}^{2J})^m$ for fixed \mathbf{k}, \vec{k} is defined by

$$|\chi^{(m)}(\mathbf{k}, \vec{k})| = \sup_{|\mathbf{x}_j| \leq 1} |\chi^{(m)}(\mathbf{k}, \vec{k})(\mathbf{x}_1, \dots, \mathbf{x}_m)|, \quad (3.12)$$

where $|\mathbf{x}|$ is the Euclidean norm.

Since $\chi_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})$ are smooth if $\mathbf{k} \notin \sigma_{\text{bc}}$, the following relation holds:

$$\max_{|\mathbf{k} \pm \mathbf{k}_{*l}| \leq \pi_0, l=1, \dots, N} |\nabla \chi_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})| \leq C'_\chi \quad (3.13)$$

if $\mathbf{k}_{*l} \notin \sigma_{\text{bc}}$, π_0 is defined by (3.3), gradient is with respect to \mathbf{k} . The case, where $\chi^{(m)}(\mathbf{k}, \vec{k})$ depend on small ϱ or, more generally, on ϱ^q , $q > 0$, can be treated similarly, see [7].

3.3. Resonance invariant nk -spectrum.

In this section, being given the dispersion relations $\omega_n(\mathbf{k}) \geq 0$, $n \in \{1, \dots, J\}$, we consider resonance properties of nk -spectra S and the corresponding k -spectra K_S as defined in Definition 2.7, i.e.,

$$\begin{aligned} S &= \{(n_l, \mathbf{k}_{*l}), l = 1, \dots, N\} \subset \Sigma = \{1, \dots, J\} \times \mathbb{R}^d, \\ K_S &= \{\mathbf{k}_{*l}, l = 1, \dots, |K_S|\}. \end{aligned} \quad (3.14)$$

We precede the formal description of the *resonance invariance* (see Definition 3.8) with the following guiding physical picture. Initially, at $\tau = 0$, the wave is a multi-wavepacket composed of modes from a small vicinity of the nk -spectrum S . As the wave evolves according to (2.1) the polynomial nonlinearity inevitably involves a larger set of modes $[S]_{\text{out}} \supseteq S$, but not all modes in $[S]_{\text{out}}$ are “equal” in developing significant amplitudes. The qualitative picture is that whenever certain interaction phase function (see (4.23) below) is not zero, the fast time oscillations weaken effective nonlinear mode interaction, and the energy transfer from the original modes in S to relevant modes from $[S]_{\text{out}}$, keeping their magnitudes vanishingly small as $\beta, \varrho \rightarrow 0$. There is a smaller set of modes $[S]_{\text{out}}^{\text{res}}$ which can interact with modes from S rather effectively and develop significant amplitudes. Now,

$$\text{if } [S]_{\text{out}}^{\text{res}} \subseteq S, \text{ then } S \text{ is called resonance invariant.} \quad (3.15)$$

In simpler situations, the resonance invariance conditions turns into the well-known in nonlinear optics phase and frequency matching conditions. For instance, if S contains (n_0, \mathbf{k}_{*l_0}) and the dispersion relations allow for the second harmonic generation in another band n_1 so that $2\omega_{n_0}(\mathbf{k}_{*l_0}) = \omega_{n_1}(2\mathbf{k}_{*l_0})$, then for S to be resonance invariant it must contain $(n_1, 2\mathbf{k}_{*l_0})$ too.

Let us turn now to the rigorous constructions. First we introduce the necessary notation. Let $m \geq 2$ be an integer, $\vec{l} = (l_1, \dots, l_m)$, $l_j \in \{1, \dots, N\}$ be an integer vector from $\{1, \dots, N\}^m$ and $\vec{\zeta} = (\zeta^{(1)}, \dots, \zeta^{(m)})$,

$\zeta^{(j)} \in \{+1, -1\}$ be a binary vector from $\{+1, -1\}^m$. Note that a pair $(\vec{\zeta}, \vec{l})$ naturally labels a sample string of the length m composed of elements $(\zeta^{(j)}, n_{l_j}, \mathbf{k}_{*l_j})$ from the set $\{+1, -1\} \times S$. Let us introduce the sets

$$\begin{aligned} \Lambda &= \{(\zeta, l) : l \in \{1, \dots, N\}, \zeta \in \{+1, -1\}\}, \\ \Lambda^m &= \{\vec{\lambda} = (\lambda_1, \dots, \lambda_m), \lambda_j \in \Lambda, j = 1, \dots, m\}. \end{aligned} \quad (3.16)$$

There is a natural one-to-one correspondence between Λ^m and $\{-1, 1\}^m \times \{1, \dots, N\}^m$, and we write, exploiting this correspondence,

$$\begin{aligned} \vec{\lambda} = ((\zeta', l_1), \dots, (\zeta^{(m)}, l_m)) &= (\vec{\zeta}, \vec{l}), \vec{\vartheta} \in \{-1, 1\}^m, \\ \vec{l} &\in \{1, \dots, N\}^m \text{ for } \vec{\lambda} \in \Lambda^m. \end{aligned} \quad (3.17)$$

Let us introduce the linear combination

$$\varkappa_m(\vec{\lambda}) = \varkappa_m(\vec{\zeta}, \vec{l}) = \sum_{j=1}^m \zeta^{(j)} \mathbf{k}_{*l_j} \text{ with } \zeta^{(j)} \in \{+1, -1\}, \quad (3.18)$$

and let $[S]_{K, \text{out}}$ be the set of all its values as $\mathbf{k}_{*l_j} \in K_S$, $\vec{\lambda} \in \Lambda^m$, namely

$$[S]_{K, \text{out}} = \bigcup_{m \in \mathfrak{M}_F} \bigcup_{\vec{\lambda} \in \Lambda^m} \{\varkappa_m(\vec{\lambda})\}. \quad (3.19)$$

We call $[S]_{K, \text{out}}$ *output k -spectrum* of K_S and assume that

$$[S]_{K, \text{out}} \cap \sigma_{\text{bc}} = \emptyset. \quad (3.20)$$

We also define the *output nk -spectrum* of S by

$$[S]_{\text{out}} = \{(n, \mathbf{k}) \in \{1, \dots, J\} \times \mathbb{R}^d : n \in \{1, \dots, J\}, \mathbf{k} \in [S]_{K, \text{out}}\}. \quad (3.21)$$

We introduce the following functions:

$$\Omega_{1, m}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^m \zeta^{(j)} \omega_{l_j}(\mathbf{k}_{*l_j}), \quad (3.22)$$

$$\vec{k}_* = (\mathbf{k}_{*1}, \dots, \mathbf{k}_{*|K_S|}), \text{ where } \mathbf{k}_{*l_j} \in K_S,$$

$$\Omega(\zeta, n, \vec{\lambda})(\mathbf{k}_{**}, \vec{k}_*) = -\zeta \omega_n(\mathbf{k}_{**}) + \Omega_{1, m}(\vec{\lambda})(\vec{k}_*), \quad (3.23)$$

where $\zeta = \pm 1$, $m \in \mathfrak{M}_F$ as in (3.7). We introduce these functions to apply later to phase functions (4.23).

Now we introduce the *resonance equation*

$$\Omega(\zeta, n, \vec{\lambda})(\zeta \varkappa_m(\vec{\lambda}), \vec{k}_*) = 0, \vec{l} \in \{1, \dots, N\}^m, \vec{\zeta} \in \{-1, 1\}^m, \quad (3.24)$$

denoting by $P(S)$ the set of its solutions $(m, \zeta, n, \vec{\lambda})$. Such a solution is called S -internal if

$$(n, \zeta \varkappa_m(\vec{\lambda})) \in S, \text{ i.e., } n = n_{l_0}, \zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{*l_0}, l_0 \in \{1, \dots, N\}, \quad (3.25)$$

and we denote the corresponding $l_0 = I(\vec{\lambda})$. We also denote by $P_{\text{int}}(S) \subset P(S)$ the set of all S -internal solutions to (3.24).

Now we consider the simplest solutions to (3.24) which play an important role. Keeping in mind that the string \vec{l} can contain several copies of a single value l , we can recast the sum in (3.22) as follows:

$$\Omega_{1,m}(\vec{\lambda}) = \Omega_{1,m}(\vec{\zeta}, \vec{l}) = \sum_{l=1}^N \delta_l \omega_l(\mathbf{k}_{*l}),$$

$$\text{where } \delta_l = \begin{cases} \sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} & \text{if } \vec{l}^{-1}(l) \neq \emptyset, \\ 0 & \text{if } \vec{l}^{-1}(l) = \emptyset, \end{cases} \quad (3.26)$$

$$\vec{l}^{-1}(l) = \{j \in \{1, \dots, m\} : l_j = l, \}, \vec{l} = (l_1, \dots, l_m), 1 \leq l \leq N.$$

Definition 3.6 (universal solutions). We call a solution $(m, \zeta, n, \vec{\lambda}) \in P(S)$ of (3.24) *universal* if it has the following properties: (i) only a single coefficient out of all δ_l in (3.26) is nonzero, namely for some I_0 we have $\delta_{I_0} = \pm 1$ and $\delta_l = 0$ for $l \neq I_0$; (ii) $n = n_{I_0}$ and $\zeta = \delta_{I_0}$.

We denote the set of universal solutions to (3.24) by $P_{\text{univ}}(S)$. A justification for calling such a solution *universal* comes from the fact that if a solution is a universal solution for one \vec{k}_* it is a solution for any other \vec{k}_* . Note that a universal solution is an S -internal solution with $I(\vec{\lambda}) = I_0$ implying

$$P_{\text{univ}}(S) \subseteq P_{\text{int}}(S). \quad (3.27)$$

Indeed, observe that for δ_l as in (3.26)

$$\varkappa_m(\vec{\lambda}) = \varkappa_m(\vec{\zeta}, \vec{l}) = \sum_{j=1}^m \zeta^{(j)} \mathbf{k}_{*l_j} = \sum_{l=1}^N \delta_l \mathbf{k}_{*l} \quad (3.28)$$

implying $\varkappa_m(\vec{\lambda}) = \delta_{I_0} \mathbf{k}_{*I_0}$ and $\zeta \varkappa_m(\vec{\lambda}) = \delta_{I_0}^2 \mathbf{k}_{*I_0} = \mathbf{k}_{*I_0}$. Then Equation (3.24) is obviously satisfied and $(n, \zeta \varkappa_m(\vec{\lambda})) = (n_{I_0}, \mathbf{k}_{*I_0}) \in S$.

Example 3.7 (universal solutions). Suppose there is just a single band, i.e., $J = 1$, a symmetric dispersion relation $\omega_1(-\mathbf{k}) = \omega_1(\mathbf{k})$, a cubic nonlinearity F with $\mathfrak{M}_F = \{3\}$. We take the nk -spectrum $S =$

$\{(1, \mathbf{k}_*), (1, -\mathbf{k}_*)\}$, i.e., $N = 2$ and $\mathbf{k}_{*1} = \mathbf{k}_*$, $\mathbf{k}_{*2} = -\mathbf{k}_*$. This example is typical for two counterpropagating waves. Then

$$\Omega_{1,3}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^3 \zeta^{(j)} \omega_{l_j}(\mathbf{k}_{*l_j}) = (\delta_1 + \delta_2) \omega_1(\mathbf{k}_*)$$

and

$$\varkappa_m(\vec{\lambda}) = \sum_{j=1}^m \zeta^{(j)} \mathbf{k}_{*l_j} = \delta_1 \mathbf{k}_{*1} + \delta_2 \mathbf{k}_{*2} = (\delta_1 - \delta_2) \mathbf{k}_*,$$

where we use the notation (3.26). The universal solution set has the form $P_{\text{univ}}(S) = \{(3, \zeta, 1, \vec{\lambda}) : \vec{\lambda} \in \Lambda_\zeta, \zeta = \pm\}$, where Λ_+ consists of vectors $(\lambda_1, \lambda_2, \lambda_3)$ of the form $((+, 1), (-, 1), (+, 1))$, $((+, 1), (-, 1), (+, 2))$, $((+, 2), (-, 2), (+, 1))$, $((+, 2), (-, 2), (+, 2))$, and vectors obtained from the listed ones by permutations of coordinates $\lambda_1, \lambda_2, \lambda_3$. The solutions from $P_{\text{int}}(S)$ have to satisfy $|\delta_1 - \delta_2| = 1$ and $|\delta_1 + \delta_2| = 1$, which is possible only if $\delta_1 \delta_2 = 0$. Since $\zeta = \delta_1 + \delta_2$, we have $\zeta \varkappa_m(\vec{\lambda}) = (\delta_1^2 - \delta_2^2) \mathbf{k}_*$ and $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{*1}$ if $|\delta_1| = 1$ or $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{*2}$ if $|\delta_2| = 1$. Hence $P_{\text{int}}(S) = P_{\text{univ}}(S)$ in this case. Note that if we set $S_1 = \{(1, \mathbf{k}_*)\}$, $S_2 = \{(1, -\mathbf{k}_*)\}$, then $S = S_1 \cup S_2$, but $P_{\text{int}}(S)$ is larger than $P_{\text{int}}(S_1) \cup P_{\text{int}}(S_2)$. This can be interpreted as follows. When only modes from S_1 are excited, the modes from S_2 remain nonexcited. But when both S_1 and S_2 are excited, there is a resonance effect of S_1 onto S_2 , represented, for example, by $\vec{\lambda} = ((+, 1), (-, 1), (+, 2))$, which involves the mode $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{*2}$.

Now we are ready to define resonance invariant spectra. First, we introduce a subset $[S]_{\text{out}}^{\text{res}}$ of $[S]_{\text{out}}$ by the formula

$$[S]_{\text{out}}^{\text{res}} = \left\{ (n, \mathbf{k}_{**}) \in [S]_{\text{out}} : \mathbf{k}_{**} = \zeta^{(0)} \varkappa_m(\vec{\lambda}), m \in \mathfrak{M}_F, \right. \\ \left. \text{where } (m, \zeta, n, \vec{\lambda}) \text{ is a solution of (3.24)} \right\}, \quad (3.29)$$

calling it *resonant output spectrum* of S , and then we define

$$\text{the resonance selection operation } \mathcal{R}(S) = S \cup [S]_{\text{out}}^{\text{res}}. \quad (3.30)$$

Definition 3.8 (resonance invariant nk -spectrum). The nk -spectrum S is called *resonance invariant* if $\mathcal{R}(S) = S$ or, equivalently, $[S]_{\text{out}}^{\text{res}} \subseteq S$. The nk -spectrum S is called *universally resonance invariant* if $\mathcal{R}(S) = S$ and $P_{\text{univ}}(S) = P_{\text{int}}(S)$.

Obviously, an nk -spectrum S is resonance invariant if and only if all solutions of (3.24) are internal, i.e., $P_{\text{int}}(S) = P(S)$.

It is worth noticing that even when an nk -spectrum is not resonance invariant, often it can be easily extended to a resonance invariant one. Namely, if $\mathcal{R}^j(S) \cap \sigma_{bc} = \emptyset$ for all j , then the set

$$\mathcal{R}^\infty(S) = \bigcup_{j=1}^{\infty} \mathcal{R}^j(S) \subset \Sigma = \{1, \dots, J\} \times \mathbb{R}^d$$

is resonance invariant. In addition to that, $\mathcal{R}^\infty(S)$ is always at most countable. Usually it is finite, i.e., $\mathcal{R}^\infty(S) = \mathcal{R}^p(S)$ for a finite p , see examples below; also $\mathcal{R}^\infty(S) = S$ for generic K_S .

Example 3.9 (resonance invariant nk -spectra for quadratic nonlinearity). Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F , i.e., $\mathfrak{M}_F = \{2\}$. Let us assume that $\mathbf{k}_* \neq 0$, $\mathbf{k}_*, 2\mathbf{k}_*, \mathbf{0}$ are not band-crossing points and look at two examples. First, suppose that $2\omega_1(\mathbf{k}_*) \neq \omega_1(2\mathbf{k}_*)$ (no second harmonic generation) and $\omega_1(\mathbf{0}) \neq 0$. Let the nk -spectrum be the set $S_1 = \{(1, \mathbf{k}_*)\}$. Then S_1 is resonance invariant. Indeed, $K_{S_1} = \{\mathbf{k}_*\}$, $[S_1]_{K, \text{out}} = \{\mathbf{0}, 2\mathbf{k}_*, -2\mathbf{k}_*\}$, $[S_1]_{\text{out}} = \{(1, \mathbf{0}), (1, 2\mathbf{k}_*), (1, -2\mathbf{k}_*)\}$ and an elementary examination shows that $[S_1]_{\text{out}}^{\text{res}} = \emptyset \subset S_1$ implying $\mathcal{R}(S_1) = S_1$. For the second example let us assume $\omega_1(\mathbf{0}) \neq 0$ and $2\omega_1(\mathbf{k}_*) = \omega_1(2\mathbf{k}_*)$, i.e., the second harmonic generation is present. Here $[S_1]_{\text{out}}^{\text{res}} = \{(1, 2\mathbf{k}_*)\}$ and $\mathcal{R}(S_1) = \{(1, \mathbf{k}_*), (1, 2\mathbf{k}_*)\}$ implying $\mathcal{R}(S_1) \neq S_1$ and hence S_1 is not resonance invariant. Suppose now that $4\mathbf{k}_*, 3\mathbf{k}_* \notin \sigma_{bc}$ and $\omega_1(\mathbf{0}) \neq 0$, $\omega_1(4\mathbf{k}_*) \neq 2\omega_1(2\mathbf{k}_*)$, $\omega_1(3\mathbf{k}_*) \neq \omega_1(\mathbf{k}_*) + \omega_1(2\mathbf{k}_*)$, and let us set $S_2 = \{(1, \mathbf{k}_*), (1, 2\mathbf{k}_*)\}$. An elementary examination shows that S_2 is resonance invariant. Note that S_2 can be obtained by iterating the resonance selection operator, namely $S_2 = \mathcal{R}(\mathcal{R}(S_1))$. Note also that $P_{\text{univ}}(S_2) \neq P_{\text{int}}(S_2)$. Notice that $\omega_1(\mathbf{0}) = 0$ is a special case since $\mathbf{k} = \mathbf{0}$ is a band-crossing point, and it requires a special treatment.

Example 3.10 (resonance invariant nk -spectra for cubic nonlinearity). Let us consider the one-band case with a symmetric dispersion relation and a cubic nonlinearity that is $\mathfrak{M}_F = \{3\}$. First we take $S_1 = \{(1, \mathbf{k}_*)\}$ and assume that $\mathbf{k}_*, 3\mathbf{k}_*$ are not band-crossing points, implying $[S_1]_{K, \text{out}} = \{\mathbf{k}_*, -\mathbf{k}_*, 3\mathbf{k}_*, -3\mathbf{k}_*\}$. We have $\Omega_{1,3}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^3 \zeta^{(j)} \omega_1(\mathbf{k}_*) = \delta_1 \omega_1(\mathbf{k}_*)$ and $\varkappa_m(\vec{\lambda}) = \delta_1 \mathbf{k}_*$, where we use the notation (3.26), δ_1 takes the values $1, -1, 3, -3$. If $3\omega_1(\mathbf{k}_*) \neq \omega_1(3\mathbf{k}_*)$, then (3.24) has a solution only if $|\delta_1| = 1$ and $\delta_1 = \zeta$. Hence $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_*$ and every solution is internal. Hence $[S_1]_{\text{out}}^{\text{res}} = \emptyset$ and $\mathcal{R}(S_1) = S_1$. Now consider the case associated with

the third harmonic generation, namely $3\omega_1(\mathbf{k}_*) = \omega_1(3\mathbf{k}_*)$ and assume that $\omega_1(3\mathbf{k}_*) + 2\omega_1(\mathbf{k}_*) \neq \omega_1(5\mathbf{k}_*)$, $3\omega_1(3\mathbf{k}_*) \neq \omega_1(9\mathbf{k}_*)$, $2\omega_1(3\mathbf{k}_*) + \omega_1(\mathbf{k}_*) \neq \omega_1(7\mathbf{k}_*)$, $2\omega_1(3\mathbf{k}_*) - \omega_1(\mathbf{k}_*) \neq \omega_1(5\mathbf{k}_*)$. An elementary examination shows that the set $S_4 = \{(1, 3\mathbf{k}_*), (1, \mathbf{k}_*), (1, -\mathbf{k}_*), (1, -3\mathbf{k}_*)\}$ satisfies $\mathcal{R}(S_4) = S_4$. Consequently, a multi-wavepacket having S_4 as its resonance invariant nk -spectrum involves the third harmonic generation and, according to Theorem 2.8, it is preserved under nonlinear evolution.

The above examples indicate that, in simple cases, the conditions on \mathbf{k}_* which can make S noninvariant with respect to \mathcal{R} have a form of several algebraic equations, Hence for almost all \mathbf{k}_* such spectra S are resonance invariant. The examples also show that if we fix S and dispersion relations, then we can include S in a larger spectrum $S' = \mathcal{R}^p(S)$ using repeated application of the operation \mathcal{R} to S , and often the resulting extended nk -spectrum S' is resonance invariant. We show in the following section that an nk -spectrum S with generic K_S is universally resonance invariant.

Note that the concept of a resonance invariant nk -spectrum gives a mathematical description of such fundamental concepts of nonlinear optics as phase matching, frequency matching, four wave interaction in cubic media, and three wave interaction in quadratic media. If a multi-wavepacket has a resonance invariant spectrum, all these phenomena may take place in the internal dynamics of the multi-wavepacket, but do not lead to resonant interactions with continuum of all remaining modes.

3.4. Genericity of the nk -spectrum invariance condition.

In simpler situations, where the number of bands J and wavepackets N are not too large, the resonance invariance of an nk -spectrum can be easily verified as above in Examples 3.9, 3.10, but what one can say if J or N are large, or if the dispersion relations are not explicitly given? We show below that, in properly defined nondegenerate cases, a small variation of K_S makes S universally resonance invariant, i.e., the resonance invariance is a generic phenomenon.

Assume that the dispersion relations $\omega_n(\mathbf{k}) \geq 0$, $n \in \{1, \dots, J\}$ are given. Observe then that $\Omega_m(\zeta, n, \vec{\lambda}) = \Omega_m(\zeta, n, \vec{\lambda})(\mathbf{k}_{*1}, \dots, \mathbf{k}_{*|\vec{K}_S|})$ defined by (3.23) is a continuous function of $\mathbf{k}_{*l} \notin \sigma_{bc}$ for every $m, \zeta, n, \vec{\lambda}$.

Definition 3.11 (ω -degenerate dispersion relations). We call dispersion relations $\omega_n(\mathbf{k})$, $n = 1, \dots, J$, ω -degenerate if there exists such a point

$\mathbf{k}_* \in \mathbb{R}^d \setminus \sigma_{bc}$ that for all \mathbf{k} in a neighborhood of \mathbf{k}_* at least one of the following four conditions holds: (i) the relations are linearly dependent, namely $\sum_{n=0}^J C_n \omega_n(\mathbf{k}) = c_0$, where all C_n are integers, one of which is nonzero, and the c_0 is a constant; (ii) at least one of $\omega_n(\mathbf{k})$ is a linear function; (iii) at least one of $\omega_n(\mathbf{k})$ satisfies the equation $C\omega_n(\mathbf{k}) = \omega_n(C\mathbf{k})$ with some n and integer $C \neq \pm 1$; (iv) at least one of $\omega_n(\mathbf{k})$ satisfies the equation $\omega_n(\mathbf{k}) = \omega_{n'}(-\mathbf{k})$, where $n' \neq n$.

Note that the fulfillment of any of four conditions in Definition 3.11 makes impossible turning some non resonance invariant sets into resonance invariant ones by a variation of \mathbf{k}_{*l} . For instance, if $\mathfrak{M}_F = \{2\}$ as in Example 3.9 and $2\omega_1(\mathbf{k}) = \omega_1(2\mathbf{k})$ for all \mathbf{k} in an open set G , then the set $\{(1, \mathbf{k}_*)\}$ with $\mathbf{k}_* \in G$ cannot be made resonance invariant by a small variation of \mathbf{k}_* . Below we formulate two theorems which show that if dispersion relations are not ω -degenerate, then a small variation of \mathbf{k}_{*l} turns non resonance invariant sets into resonance invariant; the proofs of the theorems are given in [7]

Theorem 3.12. *If $\Omega_m(\zeta, n_0, \vec{\lambda})(\mathbf{k}'_{*1}, \dots, \mathbf{k}'_{*|K_S|}) = 0$ on a cylinder G in $(\mathbb{R}^d \setminus \sigma_{bc})^{|K_S|}$ which is a product of small balls $G_i \subset (\mathbb{R}^d \setminus \sigma_{bc})$ then either $(m, \zeta, n_0, \vec{\lambda}) \in P_{univ}(S)$ or dispersion relations $\omega_n(\mathbf{k})$ are ω -degenerate as in Definition 3.11.*

Theorem 3.13 (genericity of resonance invariance). *Assume that dispersion relations $\omega_n(\mathbf{k})$ are continuous and not ω -degenerate as in Definition 3.11. Let \mathcal{K}_{rinv} be a set of points $(\mathbf{k}_{*1}, \dots, \mathbf{k}_{*|K_S|})$ such that there exists a universally resonance invariant nk -spectrum S for which its k -spectrum $K_S = \{\mathbf{k}_{*1}, \dots, \mathbf{k}_{*|K_S|}\}$. Then \mathcal{K}_{rinv} is open and everywhere dense set in $(\mathbb{R}^d \setminus \sigma_{bc})^{|K_S|}$.*

4. Integrated Evolution Equation

Using the variation of constants formula, we recast the modal evolution equation (2.1) into the following equivalent integral form:

$$\hat{\mathbf{U}}(\mathbf{k}, \tau) = \int_0^\tau e^{\frac{-i(\tau-\tau')}{\epsilon} \mathbf{L}(\mathbf{k})} \hat{F}(\hat{\mathbf{U}})(\mathbf{k}, \tau) d\tau' + e^{\frac{-i\zeta\tau}{\epsilon} \mathbf{L}(\mathbf{k})} \hat{\mathbf{h}}(\mathbf{k}), \quad \tau \geq 0. \quad (4.1)$$

Then we factor $\hat{\mathbf{U}}(\mathbf{k}, \tau)$ into the slow variable $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ and the fast oscillatory term as in (2.12), namely

$$\hat{\mathbf{U}}(\mathbf{k}, \tau) = e^{-\frac{i\tau}{\varrho} \mathbf{L}(\mathbf{k})} \hat{\mathbf{u}}(\mathbf{k}, \tau), \quad \hat{\mathbf{U}}_{n,\zeta}(\mathbf{k}, \tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \zeta \omega_n(\mathbf{k})}, \quad (4.2)$$

where $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ are the modal components of $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ as in (3.4). Notice that $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ in (4.2) may depend on ϱ and (4.2) is just a change of variables and not an assumption.

Remark 4.1. Note that if $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ is a wavepacket, it is localized near its principal wavevector \mathbf{k}_* . The expansion of $\zeta \omega_n(\mathbf{k})$ near the principal wavevector $\zeta \mathbf{k}_*$ (we take $\zeta = 1$ for brevity) takes the form

$$\omega_n(\mathbf{k}) = \omega(\mathbf{k}_*) + \nabla_k \omega_n(\mathbf{k}_*)(\mathbf{k} - \mathbf{k}_*) + \frac{1}{2} \nabla_k^2 \omega_n(\mathbf{k}_*)(\mathbf{k} - \mathbf{k}_*)^2 + \dots$$

To discuss the impact of the change of variables (4.2), we make the change of variables $\mathbf{k} - \mathbf{k}_* = \xi$. The change of variables (4.2)

$$\begin{aligned} \hat{\mathbf{U}}_{n,+}(\mathbf{k}, \tau) &= \hat{\mathbf{u}}_{n,+}(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \omega_n(\mathbf{k})} \\ &= \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \omega_n(\mathbf{k}_*)} e^{-\frac{i\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)(\mathbf{k} - \mathbf{k}_*)} e^{-\frac{i\tau}{\varrho} (\frac{1}{2} \nabla_k^2 \omega_n(\mathbf{k}_*)(\mathbf{k} - \mathbf{k}_*)^2 + \dots)} \\ &= \hat{\mathbf{u}}_{n,+}(\mathbf{k}_* + \xi, \tau) e^{-\frac{i\tau}{\varrho} \zeta \omega_n(\mathbf{k}_*)} e^{-\frac{i\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*) \xi} e^{-\frac{i\tau}{\varrho} R(\xi)}, \end{aligned} \quad (4.3)$$

$$R(\xi) = \omega_n(\mathbf{k}) - \omega_n(\mathbf{k}_*) - \nabla_k \omega_n(\mathbf{k}_*)(\mathbf{k} - \mathbf{k}_*) = \frac{1}{2} \nabla_k^2 \omega_n(\mathbf{k}_*)(\xi)^2 + \dots \quad (4.4)$$

has the first factor $e^{-\frac{i\tau}{\varrho} \omega_n(\mathbf{k}_*)}$ responsible for fast time oscillations of $\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k}, \tau)$ and $\mathbf{U}_{n,\zeta}(\mathbf{r}, \tau)$. The second factor $e^{-\frac{i\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*) \xi}$ is responsible for the spatial shifts of the inverse Fourier transform by $\frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)$. Since the shifts are time dependent, they cause the rectilinear movement of the wavepacket $\mathbf{U}_{n,\zeta}(\mathbf{r}, \tau)$ with the group velocity $\frac{1}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)$, the third factor is responsible for dispersion effects. Hence the change of variables (4.2) effectively introduces the moving coordinate frame for $\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k}, \tau)$ for every \mathbf{k} and in this coordinate frame $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ has zero group velocity and does not have high-frequency time oscillations. The following proposition shows that if $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ is a wavepacket with a constant position, $\hat{\mathbf{U}}_{n,+}(\mathbf{k}, \tau)$ is a particle wavepacket in the sense of Definition with position which moves with a constant velocity.

Proposition 4.2. *Let $\hat{\mathbf{u}}_l(\mathbf{k}, \tau)$ be for every $\tau \in [0, \tau_*]$ a particle wavepacket in the sense of Definition 2.2 with nk -pair (n, \mathbf{k}_*) , regularity s , and position $\mathbf{r}_* \in \mathbb{R}^d$ which does not depend on τ . Assume also that the constants C_1 in (2.33) and C, C' in (2.27) and (2.30) do not depend on τ . Let $\hat{\mathbf{U}}_l(\mathbf{k}, \tau)$ be defined in terms of $\hat{\mathbf{u}}_l(\mathbf{k}, \tau)$ by (4.2). Assume that (2.48)*

holds. Then $\hat{\mathbf{U}}_l(\mathbf{k}, \tau)$ for every $\tau \in [0, \tau_*]$ is a particle wavepacket in the sense of Definition 2.2 with nk -pair (n, \mathbf{k}_*) , regularity s , and τ -dependent position $\mathbf{r}_* + \frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*) \in \mathbb{R}^d$.

PROOF. The wavepacket $\hat{\mathbf{u}}_l(\mathbf{k}, \tau)$ involves two components $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$, $\zeta = \pm 1$ for which (2.29) holds

$$\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) = \Psi(\beta^{1-\varepsilon}/2, \zeta \mathbf{k}_*; \mathbf{k}) \Pi_{n,\zeta}(\mathbf{k}) \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau), \quad (4.5)$$

By (4.2),

$$\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k}, \tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \zeta \omega_n(\mathbf{k})}.$$

According to Definition 2.1, the multiplication by a scalar bounded continuous function $e^{-\frac{i\tau}{\varrho} \zeta \omega_n(\mathbf{k})}$ may only change the constant C' in (2.30). Therefore, it transforms wavepackets into wavepackets. To check that $\hat{\mathbf{U}}_l(\mathbf{k}, \tau)$ is a particle-like wavepacket, we consider (2.33) with $\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k})$ replaced by $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \zeta \omega_n(\mathbf{k})}$ and \mathbf{r}_* replaced by $\mathbf{r}_* + \frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)$. We consider for brevity $\hat{\mathbf{u}}_n(\mathbf{k}, \tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ with $\zeta = 1$, the case $\zeta = -1$ is similar,

$$\begin{aligned} & \int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i(\mathbf{r}_* + \frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)) \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \omega_n(\mathbf{k})})| d\mathbf{k} \\ &= \int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i(\mathbf{r}_* + \frac{\tau}{\varrho} \nabla_k \omega_n(\mathbf{k}_*)) \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} \omega_n(\mathbf{k})} e^{\frac{i\tau}{\varrho} \omega_n(\mathbf{k}_*)})| d\mathbf{k} \\ &= \int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i\mathbf{r}_* \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau) e^{-\frac{i\tau}{\varrho} R(\mathbf{k} - \mathbf{k}_*)})| d\mathbf{k} \leq I_1 + I_2, \end{aligned}$$

where $R(\xi)$ is defined by (4.4),

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} |e^{-\frac{i\tau}{\varrho} R(\mathbf{k} - \mathbf{k}_*)} \nabla_{\mathbf{k}}(e^{i\mathbf{r}_* \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau))| d\mathbf{k}, \\ I_2 &= \int_{\mathbb{R}^d} |(e^{i\mathbf{r}_* \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau)) \nabla_{\mathbf{k}} e^{-\frac{i\tau}{\varrho} R(\mathbf{k} - \mathbf{k}_*)}| d\mathbf{k}. \end{aligned}$$

The integral I_1 is bounded uniformly in \mathbf{r}_* by $C'_1 \beta^{-1-\varepsilon}$ since $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ satisfies (2.33). Note that

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^d} |(e^{i\mathbf{r}_* \cdot \mathbf{k}} \hat{\mathbf{u}}_n(\mathbf{k}, \tau)) \nabla_{\mathbf{k}} e^{-\frac{i\tau}{\varrho} R(\mathbf{k} - \mathbf{k}_*)}| d\mathbf{k} \\ &\leq \int_{\mathbb{R}^d} |\hat{\mathbf{u}}_n(\mathbf{k}, \tau)| \frac{\tau}{\varrho} |\nabla_{\mathbf{k}} R(\mathbf{k} - \mathbf{k}_*)| d\mathbf{k}. \end{aligned}$$

Note that, according to (4.5) and (2.25), $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) \neq 0$ only if $|\mathbf{k} - \mathbf{k}_*| \leq 2\beta^{1-\varepsilon}$, and for such $\mathbf{k} - \mathbf{k}_*$ we have the Taylor remainder estimate

$$|\nabla_{\mathbf{k}} R(\mathbf{k} - \mathbf{k}_*)| \leq C\beta^{1-\varepsilon}.$$

Therefore, $I_2 \leq C'\beta^{1-\varepsilon}/\varrho$ and

$$I_1 + I_2 \leq C'\beta^{1-\varepsilon}/\varrho.$$

Using (2.48), we conclude that this inequality implies (2.33) for $\hat{\mathbf{U}}_l(\mathbf{k}, \tau)$. Therefore, it is a particle-like wavepacket. \square

From (4.1) and (4.2) we obtain the following *integrated evolution equation* for $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{k}, \tau)$, $\tau \geq 0$:

$$\hat{\mathbf{u}}(\mathbf{k}, \tau) = \mathcal{F}(\hat{\mathbf{u}})(\mathbf{k}, \tau) + \hat{\mathbf{h}}(\mathbf{k}), \quad \mathcal{F}(\hat{\mathbf{u}}) = \sum_{m \in \mathfrak{M}_F} \mathcal{F}^{(m)}(\hat{\mathbf{u}}^m(\mathbf{k}, \tau)), \quad (4.6)$$

$$\mathcal{F}^{(m)}(\hat{\mathbf{u}}^m)(\mathbf{k}, \tau) = \int_0^\tau e^{\frac{i\tau'}{e}\mathbf{L}(\mathbf{k})} \hat{F}_m((e^{-\frac{i\tau'}{e}\mathbf{L}(\cdot)} \hat{\mathbf{u}})^m)(\mathbf{k}, \tau') d\tau', \quad (4.7)$$

where \hat{F}_m are defined by (3.7) and (3.9) in terms of the susceptibilities $\chi^{(m)}$, and $\mathcal{F}^{(m)}$ are bounded as in the following lemma.

Recall that the spaces $L^{1,a}$ are defined by formula (2.17). Below we formulate basic properties of these spaces. Recall the Young inequality

$$\|\hat{\mathbf{u}} * \hat{\mathbf{v}}\|_{L^1} \leq \|\hat{\mathbf{u}}\|_{L^1} \|\hat{\mathbf{v}}\|_{L^1}. \quad (4.8)$$

This inequality implies the boundedness of convolution in $L^{1,a}$, namely the following lemma holds.

Lemma 4.3. *Let $\hat{H}_1, \hat{H}_2 \in L^{1,a}$ be two scalar functions, $a \geq 0$. Let*

$$\hat{H}_3(\mathbf{k}) = \int_{\mathbb{R}^d} \hat{H}_1(\mathbf{k} - \mathbf{k}') \hat{H}_2(\mathbf{k}') d\mathbf{k}'.$$

Then

$$\|\hat{H}_3(\mathbf{k})\|_{L^{1,a}} \leq \|\hat{H}_1(\mathbf{k})\|_{L^{1,a}} \|\hat{H}_2(\mathbf{k})\|_{L^{1,a}}. \quad (4.9)$$

PROOF. We have

$$\begin{aligned} (1 + |\mathbf{k}|)^a |\hat{H}_3(\mathbf{k})| &\leq \sup_{\mathbf{k}', \mathbf{k}''} \frac{(1 + |\mathbf{k}' + \mathbf{k}''|)^a}{(1 + |\mathbf{k}'|)^a (1 + |\mathbf{k}''|)^a} \\ &\times \int_{\mathbb{R}^d} (1 + |\mathbf{k} - \mathbf{k}'|)^a |\hat{H}_1(\mathbf{k} - \mathbf{k}')| (1 + |\mathbf{k}'|)^a |\hat{H}_2(\mathbf{k}')| d\mathbf{k}'. \end{aligned}$$

Obviously,

$$\frac{1 + |\mathbf{k}' + \mathbf{k}''|}{(1 + |\mathbf{k}'|)(1 + |\mathbf{k}''|)} \leq \frac{(1 + |\mathbf{k}'| + |\mathbf{k}''|)}{(1 + |\mathbf{k}'|)(1 + |\mathbf{k}''|)} \leq 1.$$

Applying the Young inequality (4.8), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} (1 + |\mathbf{k}|)^a \left| \hat{H}_3(\mathbf{k}) \right| d\mathbf{k} \\ & \leq \int_{\mathbb{R}^d} (1 + |\mathbf{k}'|) \left| \hat{H}_1(\mathbf{k}') \right| d\mathbf{k}' \int_{\mathbb{R}^d} (1 + |\mathbf{k}''|) \left| \hat{H}_2(\mathbf{k}'') \right| d\mathbf{k}''. \end{aligned}$$

Using (2.18), we obtain (4.9). □

Using Lemma 4.3, we derive the boundedness of integral operators $\mathcal{F}^{(m)}$.

Lemma 4.4 (boundedness of multilinear operators). *Operator $\mathcal{F}^{(m)}$ defined by (3.9), (4.7) is bounded from $E_a = C([0, \tau_*], L^{1,a})$ into $C^1([0, \tau_*], L^{1,a})$, $a \geq 0$, and*

$$\|\mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \dots \hat{\mathbf{u}}_m)\|_{E_a} \leq \tau_* \|\chi^{(m)}\| \prod_{j=1}^m \|\hat{\mathbf{u}}_j\|_{E_a}, \quad (4.10)$$

$$\|\partial_\tau \mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \dots \hat{\mathbf{u}}_m)\|_{E_a} \leq \|\chi^{(m)}\| \prod_j \|\hat{\mathbf{u}}_j\|_{E_a}. \quad (4.11)$$

PROOF. Notice that since $\mathbf{L}(\mathbf{k})$ is Hermitian, $\|\exp\{-i\mathbf{L}(\mathbf{k})\frac{\tau_1}{\varrho}\}\| = 1$. Using the inequality (4.9) together with (3.9), (4.7), we obtain

$$\begin{aligned} & \|\mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \dots \hat{\mathbf{u}}_m)(\cdot, \tau)\|_{L^{1,a}} \\ & \leq \sup_{\mathbf{k}, \vec{k}} |\chi^{(m)}(\mathbf{k}, \vec{k})| \int_{\mathbb{R}^d} \int_0^\tau \int_{\mathbb{D}_m} |(1 + |\mathbf{k}'|)^a \hat{\mathbf{u}}_1(\mathbf{k}')| \dots \\ & \quad \times \left| (1 + |\mathbf{k}^{(m)}|)^a \hat{\mathbf{u}}_m(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \right| d\mathbf{k}' \dots d\mathbf{k}^{(m-1)} d\tau_1 d\mathbf{k} \\ & \leq \|\chi^{(m)}\| \int_0^\tau \|\hat{\mathbf{u}}_1(\tau_1)\|_{L^{1,a}} \dots \|\hat{\mathbf{u}}_m(\tau_1)\|_{L^{1,a}} d\tau_1 \\ & \leq \tau_* \|\chi^{(m)}\| \|\hat{\mathbf{u}}_1\|_{E_a} \dots \|\hat{\mathbf{u}}_m\|_{E_a}, \end{aligned}$$

proving (4.10). A similar estimate produces (4.11). □

Equation (4.6) can be recast as the following abstract equation in a Banach space:

$$\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}, \quad \hat{\mathbf{u}}, \hat{\mathbf{h}} \in E_a, \quad (4.12)$$

and it readily follows from Lemma 4.4 that $\mathcal{F}(\hat{\mathbf{u}})$ has the following properties.

Lemma 4.5. *The operator $\mathcal{F}(\hat{\mathbf{u}})$ defined by (4.6), (4.7) satisfies the Lipschitz condition*

$$\|\mathcal{F}(\hat{\mathbf{u}}_1) - \mathcal{F}(\hat{\mathbf{u}}_2)\|_{E_a} \leq \tau_* C_F \|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2\|_{E_a} \quad (4.13)$$

where $C_F \leq C_\chi C(R)$, $C(R)$ depends only on m_F and R , if $\|\hat{\mathbf{u}}_1\|_{E_a}, \|\hat{\mathbf{u}}_2\|_{E_a} \leq 2R$, with C_χ as in (3.11).

We also use the following form of the contraction principle.

Lemma 4.6 (contraction principle). *Consider the equation*

$$\mathbf{x} = \mathcal{F}(\mathbf{x}) + \mathbf{h}, \quad \mathbf{x}, \mathbf{h} \in B, \quad (4.14)$$

where B is a Banach space, \mathcal{F} is an operator in B . Suppose that for some constants $R_0 > 0$ and $0 < q < 1$

$$\|\mathbf{h}\| \leq R_0, \quad \|\mathcal{F}(\mathbf{x})\| \leq R_0 \quad \text{if } \|\mathbf{x}\| \leq 2R_0, \quad (4.15)$$

$$\|\mathcal{F}(\mathbf{x}_1) - \mathcal{F}(\mathbf{x}_2)\| \leq q \|\mathbf{x}_1 - \mathbf{x}_2\| \quad \text{if } \|\mathbf{x}_1\|, \|\mathbf{x}_2\| \leq 2R_0. \quad (4.16)$$

Then there exists a unique solution \mathbf{x} to Equation (4.14) such that $\|\mathbf{x}\| \leq 2R_0$. Let $\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \leq R_0$. Then two corresponding solutions $\mathbf{x}_1, \mathbf{x}_2$ satisfy

$$\|\mathbf{x}_1\|, \|\mathbf{x}_2\| \leq 2R_0, \quad \|\mathbf{x}_1 - \mathbf{x}_2\| \leq (1 - q)^{-1} \|\mathbf{h}_1 - \mathbf{h}_2\|. \quad (4.17)$$

Let $\mathbf{x}_1, \mathbf{x}_2$ be two solutions of correspondingly two equations of the form (4.14) with $\mathcal{F}_1, \mathbf{h}_1$ and $\mathcal{F}_2, \mathbf{h}_2$. Assume that $\mathcal{F}_1(\mathbf{u})$ satisfies (4.15), (4.16) with a Lipschitz constant $q < 1$ and that $\|\mathcal{F}_1(\mathbf{x}) - \mathcal{F}_2(\mathbf{x})\| \leq \delta$ for $\|\mathbf{x}\| \leq 2R_0$. Then

$$\|\mathbf{x}_1 - \mathbf{x}_2\| \leq (1 - q)^{-1} (\delta + \|\mathbf{h}_1 - \mathbf{h}_2\|). \quad (4.18)$$

Lemma 4.5 and the contraction principle as in Lemma 4.6 imply the following existence and uniqueness theorem.

Theorem 4.7. *Let $\|\hat{\mathbf{h}}\|_{E_a} \leq R$, and let $\tau_* < 1/C_F$, where C_F is a constant from Lemma 4.5. Then Equation (4.6) has a solution $\hat{\mathbf{u}} \in E_a = C([0, \tau_*], L^{1,\alpha})$ which satisfies $\|\hat{\mathbf{u}}\|_{E_a} \leq 2R$, and such a solution is unique. Hence the solution operator $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ is defined on the ball $\|\hat{\mathbf{h}}\|_{E_a} \leq R$.*

The following existence and uniqueness theorem is a consequence of Theorem 4.7.

Theorem 4.8. *Let $a \geq 0$, (2.1) satisfy (3.11) and $\hat{\mathbf{h}} \in L^{1,a}(\mathbb{R}^d)$, $\|\hat{\mathbf{h}}\|_{L^{1,a}} \leq R$. Then there exists a unique solution $\hat{\mathbf{u}}$ to the modal evolution equation (2.1) in the function space $C^1([0, \tau_*], L^{1,a})$, $\|\hat{\mathbf{u}}\|_{E_a} + \|\partial\tau\hat{\mathbf{u}}\|_{E_a} \leq R_1(R)$. The number τ_* depends on R and C_χ .*

Using (2.20) and applying the inverse Fourier transform, we readily obtain the existence of an F -solution of (1.1) in $C^1([0, \tau_*], L^\infty(\mathbb{R}^d))$ from the existence of the solution of Equation (2.1) in $C^1([0, \tau_*], L^1)$. The existence of F -solutions with $[a]$ bounded spatial derivatives ($[a]$ being an integer part of a) follows from the solvability in $C^1([0, \tau_*], L^{1,a})$.

Let us recast now the system (4.6), (4.7) into modal components using the projections $\Pi_{n,\zeta}(\mathbf{k})$ as in (2.9). The first step to introduce modal susceptibilities $\chi_{n,\zeta,\vec{\xi}}^{(m)}$ having one-dimensional range in \mathbb{C}^{2J} and vanishing if one of its arguments $\hat{\mathbf{u}}_j$ belongs to a $(2J - 1)$ -dimensional linear subspace in \mathbb{C}^{2J} (the j th null-space of $\chi_{n,\zeta,\vec{\xi}}^{(m)}$) as follows.

Definition 4.9 (elementary susceptibilities). Let

$$\vec{\xi} = (\vec{n}, \vec{\zeta}) \in \{1, \dots, J\}^m \times \{-1, 1\}^m = \Xi^m, (n, \zeta) \in \Xi, \quad (4.19)$$

and let $\chi^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})]$ be m -linear symmetric tensor (susceptibility) as in (3.9).

We introduce *elementary susceptibilities* $\chi_{n,\zeta,\vec{\xi}}^{(m)}(\mathbf{k}, \vec{k}) : (\mathbb{C}^{2J})^m \rightarrow \mathbb{C}^{2J}$ as m -linear tensors defined for almost all \mathbf{k} and $\vec{k} = (\mathbf{k}', \dots, \mathbf{k}^{(m)})$ by the following formula:

$$\begin{aligned} & \chi_{n,\zeta,\vec{\xi}}^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})] \\ &= \chi_{n,\zeta,\vec{n},\vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})] = \Pi_{n,\zeta}(\mathbf{k}) \chi^{(m)}(\mathbf{k}, \vec{k}) \\ & \times [(\Pi_{n_1,\zeta'}(\mathbf{k}') \hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \Pi_{n_m,\zeta^{(m)}}(\mathbf{k}^{(m)}) \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})]. \end{aligned} \quad (4.20)$$

Then using (3.5) and the *elementary susceptibilities* (4.20), we get

$$\begin{aligned} & \chi^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})] \\ &= \sum_{n,\zeta} \sum_{\vec{\xi}} \chi_{n,\zeta,\vec{\xi}}^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{u}}_1(\mathbf{k}'), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)})]. \end{aligned} \quad (4.21)$$

Consequently, the modal components $\mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}$ of the operators $\mathcal{F}^{(m)}$ in (4.7) are m -linear *oscillatory integral operators* defined in terms of the elementary susceptibilities (4.21) as follows.

Definition 4.10 (interaction phase). Using the notation from (3.9), we introduce for $\vec{\xi} = (\vec{n}, \vec{\zeta}) \in \Xi^m$ the operator

$$\begin{aligned} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}(\hat{\mathbf{u}}_1 \dots \hat{\mathbf{u}}_m)(\mathbf{k}, \tau) &= \int_0^\tau \int_{\mathbb{D}_m} \exp \left\{ i\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} \\ &\times \chi_{n,\zeta,\vec{\xi}}^{(m)}(\mathbf{k}, \vec{k}) \left[\hat{\mathbf{u}}_1(\mathbf{k}', \tau_1), \dots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), \tau_1) \right] \tilde{d}^{(m-1)d} \vec{k} d\tau_1 \end{aligned} \quad (4.22)$$

with the *interaction phase function* φ defined by

$$\begin{aligned} \varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}, \vec{k}) &= \varphi_{n,\zeta,\vec{n},\vec{\zeta}}(\mathbf{k}, \vec{k}) \\ &= \zeta \omega_n(\zeta \mathbf{k}) - \zeta' \omega_{n_1}(\zeta' \mathbf{k}') - \dots - \zeta^{(m)} \omega_{n_m}(\zeta^{(m)} \mathbf{k}^{(m)}), \end{aligned} \quad (4.23)$$

$$\mathbf{k}^{(m)} = \mathbf{k}^{(m)}(\mathbf{k}, \vec{k}),$$

where $\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})$ is defined by (3.10).

Using $\mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}$ in (4.22), we recast $\mathcal{F}^{(m)}(\mathbf{u}^m)$ in the system (4.6)-(4.7) as follows:

$$\mathcal{F}^{(m)}[\hat{\mathbf{u}}_1 \dots, \hat{\mathbf{u}}_m](\mathbf{k}, \tau) = \sum_{n,\zeta,\vec{\xi}} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}[\hat{\mathbf{u}}_1 \dots \hat{\mathbf{u}}_m](\mathbf{k}, \tau) \quad (4.24)$$

yielding the following system for the modal components $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau)$ as in (2.9):

$$\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}, \tau) = \sum_{m \in \mathfrak{M}_F} \sum_{\vec{\xi} \in \Xi^m} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}(\hat{\mathbf{u}}^m)(\mathbf{k}, \tau) + \hat{\mathbf{h}}_{n,\zeta}(\mathbf{k}), \quad (n, \zeta) \in \Xi. \quad (4.25)$$

5. Wavepacket Interaction System

The wavepacket preservation property of the nonlinear evolutionary system in any of its forms (1.1), (2.1), (4.6), (4.12), (4.25) is not easy to see directly. It turns out though that dynamics of wavepackets is well described by a system in a larger space E^{2N} based on the original equation (4.6) in the space E . We call it *wavepacket interaction system*, which is useful in three ways: (i) the wavepacket preservation is quite easy to see and verify; (ii) it can be used to prove the wavepacket preservation for the original nonlinear problem; (iii) it can be used to study more subtle properties of the original problem, such as the NLS approximation. We start with the system (4.6), where $\hat{\mathbf{h}}(\mathbf{k})$ is a multi-wavepacket with

a given nk -spectrum $S = \{(\mathbf{k}_{*l}, n_l), l = 1, \dots, N\}$ as in (2.39) and a k -spectrum $K_S = \{\mathbf{k}_{*i}, i = 1, \dots, |K_S|\}$ as in (2.40). Obviously, for any l $(\mathbf{k}_{*l}, n_l) = (\mathbf{k}_{*i_l}, n_l)$ with $i_l \leq |K_S|$ and indexing $i_l = l$ for $l \leq |K_S|$ according to (2.40).

When constructing the wavepacket interaction system it is convenient to have relevant functions to be explicitly localized about the k -spectrum K_S of the initial data. We implement that by making up the following cutoff functions based on (2.24), (2.25):

$$\begin{aligned} \Psi_{i,\vartheta}(\mathbf{k}) &= \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i}, \beta^{1-\varepsilon}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \vartheta \mathbf{k}_{*i})), \\ \mathbf{k}_{*i} &\in K_S, \quad i = 1, \dots, |K_S|, \quad \vartheta = \pm, \end{aligned} \quad (5.1)$$

with ε as in Definition 2.1 and $\beta > 0$ small enough to satisfy

$$\beta^{1/2} \leq \pi_0, \quad \text{where } \pi_0 = \pi_0(S) = \frac{1}{2} \min_{\mathbf{k}_{*i} \in K_S} \text{dist} \{ \mathbf{k}_{*i}, \sigma_{bc} \}. \quad (5.2)$$

In what follows, we use the notation from (3.16) and

$$\begin{aligned} \vec{l} &= (l_1, \dots, l_m) \in \{1, \dots, N\}^m, \\ \vec{\vartheta} &= (\vartheta', \dots, \vartheta^{(m)}) \in \{-1, 1\}^m, \quad \vec{\lambda} = (\vec{l}, \vec{\vartheta}) \in \Lambda^m, \end{aligned} \quad (5.3)$$

$$\vec{n} = (n_1, \dots, n_m) \in \{1, \dots, J\}^m, \quad \vec{\zeta} \in \{-1, 1\}^m, \quad (5.4)$$

$$\vec{\xi} = (\vec{n}, \vec{\zeta}) \in \Xi^m, \quad \vec{k} = (\mathbf{k}', \dots, \mathbf{k}^{(m)}) \in \mathbb{R}^m, \quad \text{where } \Xi^m \text{ as in (4.19).}$$

Based on the above, we introduce now the *wavepacket interaction system*

$$\begin{aligned} \hat{\mathbf{w}}_{l,\vartheta}(\cdot) &= \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta}(\cdot) \mathcal{F} \left(\sum_{(l', \vartheta') \in \Lambda} \hat{\mathbf{w}}_{l', \vartheta'} \right) \\ &+ \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta}(\cdot) \hat{\mathbf{h}}, \quad (l, \vartheta) \in \Lambda, \end{aligned} \quad (5.5)$$

$$\tilde{\mathbf{w}} = (\hat{\mathbf{w}}_{1,+}, \hat{\mathbf{w}}_{1,-}, \dots, \hat{\mathbf{w}}_{N,+}, \hat{\mathbf{w}}_{N,-}) \in E^{2N}, \quad \hat{\mathbf{w}}_{l,\vartheta} \in E,$$

with $\Psi(\cdot, \vartheta \mathbf{k}_{*i}), \Pi_{n,\vartheta}$ being as in (5.1), (2.9), \mathcal{F} defined by (4.6), and the norm in E^{2N} defined based on (2.15) by the formula

$$\|\tilde{\mathbf{w}}\|_{E^{2N}} = \sum_{l,\vartheta} \|\hat{\mathbf{w}}_{l,\vartheta}\|_E, \quad E = C([0, \tau_*], L^1). \quad (5.6)$$

We also use the following concise form of the wave interaction system (5.5):

$$\tilde{\mathbf{w}} = \mathcal{F}_\Psi(\tilde{\mathbf{w}}) + \tilde{\mathbf{h}}_\Psi, \quad \text{where} \quad (5.7)$$

$$\tilde{\mathbf{h}}_\Psi = (\Psi_{i_1,+} \Pi_{n_1,+} \hat{\mathbf{h}}, \Psi_{i_1,-} \Pi_{n_1,-} \hat{\mathbf{h}}, \dots, \Psi_{i_N,+} \Pi_{n_N,+} \hat{\mathbf{h}}, \Psi_{i_N,-} \Pi_{n_N,-} \hat{\mathbf{h}}) \in E^{2N}.$$

The following lemma is analogous to Lemmas 4.4 and 4.5.

Lemma 5.1. *The polynomial operator $\mathcal{F}_\Psi(\tilde{\mathbf{w}})$ is bounded in E^{2N} , $\mathcal{F}_\Psi(\mathbf{0}) = \mathbf{0}$, and satisfies Lipschitz condition*

$$\|\mathcal{F}_\Psi(\tilde{\mathbf{w}}_1) - \mathcal{F}_\Psi(\tilde{\mathbf{w}}_2)\|_{E^{2N}} \leq C\tau_* \|\tilde{\mathbf{w}}_1 - \tilde{\mathbf{w}}_2\|_{E^{2N}}, \quad (5.8)$$

where C depends only on C_χ as in (3.11), on the degree of \mathcal{F} , and on $\|\tilde{\mathbf{w}}_1\|_{E^{2N}} + \|\tilde{\mathbf{w}}_2\|_{E^{2N}}$, and it does not depend on β and ϱ .

PROOF. We consider any operator $\mathcal{F}_{n,\zeta,\tilde{\xi}}^{(m)}(\tilde{\mathbf{w}})$ defined by (4.22) and prove its boundedness and the Lipschitz property as in Lemma 4.4 using the inequality $\left| \exp \left\{ i\varphi_{n,\zeta,\tilde{\xi}} \frac{\tau_1}{\varrho} \right\} \right| \leq 1$ and inequalities (2.24), (3.11). Note that the integration in τ_1 yields the factor τ_* and consequent summation with respect to $n, \zeta, \tilde{\xi}$ yields (5.8). \square

Lemma 5.1, the contraction principle as in Lemma 4.6, and the estimate (4.11) for the time derivative yield the following statement.

Theorem 5.2. *Let $\|\tilde{\mathbf{h}}_\Psi\|_{E^{2N}} \leq R$. Then there exists $\tau_* > 0$ and $R_1(R)$ such that Equation (5.5) has a solution $\tilde{\mathbf{w}} \in E^{2N}$ which satisfies*

$$\|\tilde{\mathbf{w}}\|_{E^{2N}} + \|\partial_\tau \tilde{\mathbf{w}}\|_{E^{2N}} \leq R_1(R), \quad (5.9)$$

and such a solution is unique.

Lemma 5.3. *Every function $\hat{\mathbf{w}}_{l,\zeta}(\mathbf{k}, \tau)$ corresponding to the solution of (5.7) from E^{2N} is a wavepacket with nk -pair (\mathbf{k}_{*l}, n_l) with the degree of regularity which can be any $s > 0$.*

PROOF. Note that, according to (5.1) and (5.7), the function

$$\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau) = \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i_l}, \beta^{1-\varepsilon}) \Pi_{n_l, \vartheta} \mathcal{F}(\mathbf{k}, \tau), \quad \|\mathcal{F}(\tau)\|_{L^1} \leq C, \quad 0 \leq \tau \leq \tau_*$$

involves the factor $\Psi_{l,\vartheta}(\mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \vartheta \mathbf{k}_{*l}))$, where ε is as in Definition 2.1. Hence

$$\Pi_{n,\vartheta'} \hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau) = 0 \text{ if } n \neq n_l \text{ or } \vartheta' \neq \vartheta, \quad (5.10)$$

$$\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau) = \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i_l}, \beta^{1-\varepsilon}) \hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau), \quad (5.11)$$

$$\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau) = 0 \text{ if } |\mathbf{k} - \vartheta \mathbf{k}_{*l}| \geq \beta^{1-\varepsilon},$$

Since

$$\Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i_l}, \beta^{1-\varepsilon}) \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i_l}, \beta^{1-\varepsilon}/2) = \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i_l}, \beta^{1-\varepsilon}), \quad (5.12)$$

Definition 2.1 for $\hat{\mathbf{w}}_{l,\vartheta}$ is satisfied with $D_h = 0$ for any $s > 0$ and $C' = 0$ in (2.30). \square

Now we would like to show that if $\hat{\mathbf{h}}$ is a multi-wavepacket, then the function

$$\hat{\mathbf{w}}(\mathbf{k}, \tau) = \sum_{(l, \vartheta) \in \Lambda} \hat{\mathbf{w}}_{l, \vartheta}(\mathbf{k}, \tau) = \sum_{\lambda \in \Lambda} \hat{\mathbf{w}}_{\lambda}(\mathbf{k}, \tau) \quad (5.13)$$

constructed from a solution of (5.7) is an approximate solution of Equation (4.12) (see the notation (3.16)). We follow here the lines of [7]. We introduce

$$\Psi_{\infty}(\mathbf{k}) = 1 - \sum_{\vartheta = \pm} \sum_{i=1}^{|K_S|} \Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i}) = 1 - \sum_{\vartheta = \pm} \sum_{\mathbf{k}_{*i} \in K_S} \Psi\left(\frac{\mathbf{k} - \vartheta \mathbf{k}_{*i}}{\beta^{1-\varepsilon}}\right). \quad (5.14)$$

Expanding the m -linear operator $\mathcal{F}^{(m)}\left(\left(\sum_{l, \vartheta} \hat{\mathbf{w}}_{l, \vartheta}\right)^m\right)$ and using the notation (3.16), (3.17), we get

$$\mathcal{F}^{(m)}\left(\left(\sum_{l, \vartheta} \hat{\mathbf{w}}_{l, \vartheta}\right)^m\right) = \sum_{\vec{\lambda} \in \Lambda^m} \mathcal{F}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}}), \quad \text{where} \quad (5.15)$$

$$\vec{\mathbf{w}}_{\vec{\lambda}} = \hat{\mathbf{w}}_{\lambda_1} \dots \hat{\mathbf{w}}_{\lambda_m}, \quad \vec{\lambda} = (\lambda_1, \dots, \lambda_m) \in \Lambda^m. \quad (5.16)$$

The next statement shows that (5.13) defines an approximate solution to the integrated evolution equation (4.6).

Theorem 5.4. *Let $\hat{\mathbf{h}}$ be a multi-wavepacket with resonance invariant nk -spectrum S and regularity degree s , let $\vec{\mathbf{w}}$ be a solution of (5.7), and let $\hat{\mathbf{w}}(\mathbf{k}, \tau)$ be defined by (5.13). Let*

$$\mathbf{D}(\hat{\mathbf{w}}) = \hat{\mathbf{w}} - \mathcal{F}(\hat{\mathbf{w}}) - \hat{\mathbf{h}}. \quad (5.17)$$

Then there exists $\beta_0 > 0$ such that

$$\|\mathbf{D}(\hat{\mathbf{w}})\|_E \leq C \varrho + C \beta^s, \quad \text{if } 0 < \varrho \leq 1, \quad \beta \leq \beta_0. \quad (5.18)$$

PROOF. Let

$$\begin{aligned} \mathcal{F}^-(\hat{\mathbf{w}}) &= \left(1 - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta}\right) \mathcal{F}(\hat{\mathbf{w}}), \\ \hat{\mathbf{h}}^- &= \hat{\mathbf{h}} - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{\mathbf{h}}. \end{aligned} \quad (5.19)$$

Summation of (5.5) with respect to l, ϑ yields

$$\hat{\mathbf{w}} = \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \mathcal{F}(\hat{\mathbf{w}}) + \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{\mathbf{h}}.$$

Hence from (5.5) and (5.17) we obtain

$$\mathbf{D}(\hat{\mathbf{w}}) = \hat{\mathbf{h}}^- - \mathcal{F}^-(\hat{\mathbf{w}}). \quad (5.20)$$

Using (2.28) and (2.30), we consequently obtain

$$\begin{aligned} \|\Pi_{n_l, \vartheta} \hat{\mathbf{h}}_i\|_{L^1} &\leq C\beta^s \text{ if } n_l \neq n_i; \\ \|\Psi_{i_l, \vartheta} \hat{\mathbf{h}}_i\|_{L^1} &\leq C\beta^s \text{ if } \mathbf{k}_{*i_l} \neq \mathbf{k}_{*i}, \\ \|\hat{\mathbf{h}}^-\|_E &\leq C_1\beta^s. \end{aligned} \quad (5.21)$$

To show (5.18), it suffices to prove that

$$\|\mathcal{F}^-(\hat{\mathbf{w}})\|_E \leq C_2\varrho. \quad (5.22)$$

Obviously,

$$\mathcal{F}^-(\hat{\mathbf{w}}) = \left(1 - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta}\right) \sum_m \mathcal{F}^{(m)}(\hat{\mathbf{w}}^m). \quad (5.23)$$

Note that

$$\sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} = \sum_{\vartheta=\pm} \sum_{(n, \mathbf{k}_*) \in S} \Psi(\cdot, \vartheta \mathbf{k}_*) \Pi_{n, \vartheta}. \quad (5.24)$$

Using (3.5) and (5.14), we consequently obtain

$$\sum_{\vartheta=\pm} \sum_{(n, \mathbf{k}_*) \in \Sigma} \Psi(\cdot, \vartheta \mathbf{k}_*) \Pi_{n, \vartheta} + \Psi_\infty = 1, \quad (5.25)$$

$$\left(1 - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta}\right) = \Psi_\infty + \sum_{\vartheta=\pm} \sum_{(n, \mathbf{k}_*) \in \Sigma \setminus S} \Psi(\cdot, \vartheta \mathbf{k}_*) \Pi_{n, \vartheta} \quad (5.26)$$

with the set Σ defined in (3.14). Let us expand now $\mathcal{F}^{(m)}(\hat{\mathbf{w}}^m)$ using (5.15). According to (5.23) and (5.26), to prove (5.22) it suffices to prove that for every string $\vec{\lambda} \in \Lambda^m$ the following inequalities hold:

$$\|\Psi_\infty \Pi_{n, \vartheta} \mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})\| \leq C_3\varrho \text{ for } (n, \vartheta) \in \Lambda, \quad (5.27)$$

$$\|\Psi(\cdot, \vartheta \mathbf{k}_*) \Pi_{n, \vartheta} \mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})\| \leq C_3\varrho \text{ if } (n, \mathbf{k}_*) \in \Sigma \setminus S. \quad (5.28)$$

We use (5.10) and (5.11) to obtain the above estimates. According to (4.24),

$$\mathcal{F}^{(m)}[\tilde{\mathbf{w}}_{\vec{\lambda}}](\mathbf{k}, \tau) = \sum_{n, \zeta} \sum_{\vec{\xi}} \mathcal{F}_{n, \zeta, \vec{\xi}}^{(m)}[\hat{\mathbf{w}}_{\lambda_1} \dots \hat{\mathbf{w}}_{\lambda_m}](\mathbf{k}, \tau). \quad (5.29)$$

Note that, according to (5.10), if $\lambda_i = (l, \vartheta')$, then

$$\hat{\mathbf{w}}_{\lambda_i} = \Pi_{n, \vartheta} \hat{\mathbf{w}}_{\lambda_i} \text{ if } n = n_l \text{ and } \vartheta' = \vartheta. \quad (5.30)$$

Let us introduce the notation

$$\vec{n}(\vec{l}) = (n_{l_1}, \dots, n_{l_m}), \quad \vec{\xi}(\vec{\lambda}) = (\vec{n}(\vec{l}), \vec{\vartheta}) \text{ for } \vec{\lambda} = (\vec{l}, \vec{\vartheta}) \in \Lambda^m. \quad (5.31)$$

Since

$$\Pi_{n', \vartheta} \Pi_{n, \vartheta'} = 0 \text{ if } n \neq n' \text{ or } \vartheta' \neq \vartheta, \quad (5.32)$$

(5.30) implies

$$\begin{aligned} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}[\widehat{\mathbf{w}}_{\lambda_1} \dots \widehat{\mathbf{w}}_{\lambda_m}] &= 0 \text{ if } \vec{\xi} = (\vec{n}, \vec{\zeta}) \neq \vec{\xi}(\vec{\lambda}), \text{ and hence} \\ \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}[\widehat{\mathbf{w}}_{\vec{\lambda}}](\mathbf{k}, \tau) &= \sum_{n,\zeta} \mathcal{F}_{n,\zeta,\vec{\xi}(\vec{\lambda})}^{(m)}[\widehat{\mathbf{w}}_{\lambda_1} \dots \widehat{\mathbf{w}}_{\lambda_m}](\mathbf{k}, \tau), \end{aligned} \quad (5.33)$$

where we used the notation (3.17), (5.31). Note also that

$$\Pi_{n',\vartheta} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)} = 0 \text{ if } n' \neq n \text{ or } \vartheta \neq \zeta, \quad (5.34)$$

and hence we have nonzero $\Pi_{n',\vartheta} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}(\widehat{\mathbf{w}}_{\vec{\lambda}})$ only if

$$\vec{\xi} = \vec{\xi}(\vec{\lambda}), \quad n' = n, \quad \vartheta = \zeta. \quad (5.35)$$

By (4.22),

$$\begin{aligned} \mathcal{F}_{n,\zeta,\vec{\xi}(\vec{\lambda})}^{(m)}(\widehat{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k}, \tau) &= \int_0^\tau \int_{\mathbb{D}_m} \exp \left\{ i\varphi_{n,\zeta,\vec{\xi}(\vec{\lambda})}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} \chi_{n,\zeta,\vec{\xi}(\vec{\lambda})}^{(m)}(\mathbf{k}, \vec{k}) \\ &\times [\widehat{\mathbf{w}}_{\lambda_1}(\mathbf{k}', \tau_1), \dots, \widehat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), \tau_1)] \tilde{d}^{(m-1)d} \vec{k} d\tau_1. \end{aligned} \quad (5.36)$$

Now we use (5.11) and notice that, according to the convolution identity in (3.9),

$$\begin{aligned} |\widehat{\mathbf{w}}_{\lambda_1}(\mathbf{k}', \tau_1)| \cdot \dots \cdot |\widehat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), \tau_1)| &= 0 \\ \text{if } \left| \mathbf{k} - \sum_i \vartheta_i \mathbf{k}_{*l_i} \right| &\geq m\beta^{1-\varepsilon}. \end{aligned} \quad (5.37)$$

Hence the integral (5.36) is nonzero only if (\mathbf{k}, \vec{k}) belongs to the set

$$\begin{aligned} B_\beta = \left\{ (\mathbf{k}, \vec{k}) : |\mathbf{k}^{(i)} - \vartheta_i \mathbf{k}_{*l_i}| \leq \beta^{1-\varepsilon}, \quad i = 1, \dots, m, \right. \\ \left. \left| \mathbf{k} - \sum_i \vartheta_i \mathbf{k}_{*l_i} \right| \leq m\beta^{1-\varepsilon} \right\}. \end{aligned} \quad (5.38)$$

We prove now that if $(n, \mathbf{k}_{*i}) \notin S$, then for small β the following alternative holds:

$$\text{either } \Psi(\cdot, \vartheta \mathbf{k}_{*i}) \Pi_{n',\vartheta} \mathcal{F}_{n,\zeta,\vec{\xi}}^{(m)}(\widehat{\mathbf{w}}_{\vec{\lambda}}) = 0 \quad (5.39)$$

$$\text{or (5.35) holds and } |\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}, \vec{k})| \geq c > 0 \text{ for } (\mathbf{k}, \vec{k}) \in B_\beta. \quad (5.40)$$

Since $\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}, \vec{k})$ is smooth, in the notation (3.18) we get

$$\begin{aligned} |\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}, \vec{k}) - \varphi_{n',\zeta,\vec{\xi}}(\mathbf{k}_{**}, \vec{k}_{**})| &\leq C\beta^{1-\varepsilon} \text{ for } (\mathbf{k}, \vec{k}) \in B_\beta, \\ \vec{\vartheta} = (\vartheta_1, \dots, \vartheta_m), \quad \mathbf{k}_{**} = \zeta \sum_i \vartheta_i \mathbf{k}_{*l_i} &= \zeta \varkappa_m(\vec{\vartheta}, \vec{l}), \end{aligned} \quad (5.41)$$

Hence (5.40) holds if

$$\varphi_{n,\zeta,\xi}(\mathbf{k}_{**}, \vec{k}_*) \neq 0, \quad (5.42)$$

and, consequently, it suffices to prove that either (5.39) or (5.42) holds. Combining (5.38) with $\Psi(\mathbf{k}, \vartheta \mathbf{k}_{*i}) = 0$ for $|\mathbf{k} - \vartheta \mathbf{k}_{*i}| \geq \beta^{1-\varepsilon}$, we find that $\Psi_{i,\vartheta} \mathcal{F}^{(m)}[\vec{\mathbf{w}}_{\vec{\lambda}}]$ can be nonzero for small β only in a small neighborhood of a point $\zeta \varkappa_m(\vec{\vartheta}, \vec{l}) \in [S]_{K,\text{out}}$, and that is possible only if

$$\mathbf{k}_{**} = \zeta \varkappa_m(\vec{\vartheta}, \vec{l}) = \vartheta \mathbf{k}_{*i}, \quad \mathbf{k}_{*i} \in K_S. \quad (5.43)$$

Let us show that the equality

$$\varphi_{n,\zeta,\xi}(\mathbf{k}_{**}, \vec{k}_*) = 0 \quad (5.44)$$

is impossible for \mathbf{k}_{**} as in (5.43) and $n' = n$ as in (5.34), keeping in mind that $(n, \mathbf{k}_{*i}) \notin S$. From (3.23) and (4.23) it follows that Equation (5.44) has the form of the resonance equation (3.24). Since the nk -spectrum S is resonance invariant, in view of Definition 3.8 the resonance equation (5.44) may have a solution only if $\mathbf{k}_{**} = \mathbf{k}_{*i}$, $i = i_l$, $n = n_l$, with $(n_l, \mathbf{k}_{*i_l}) \in S$. Since $(n, \mathbf{k}_{*i}) \notin S$, that implies (5.44) does not have a solution. Hence (5.42) holds when $(n, \mathbf{k}_{*i}) \notin S$. Notice that (5.9) yields the following bounds

$$\|\hat{\mathbf{w}}_{\lambda_i}\|_E \leq R_1, \quad \|\partial_\tau \hat{\mathbf{w}}_{\lambda_i}\|_E \leq C. \quad (5.45)$$

These bounds combined with Lemma 5.5, proved below, imply that if (5.42) holds, then (5.28) holds. Now let us turn to (5.27). According to (5.14) and (5.37), the term $\Psi_\infty \Pi_{n',\vartheta} \mathcal{F}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}})$ can be nonzero only if $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{**} \notin K_S$. Since the nk -spectrum S is resonance invariant we conclude as above that the inequality (5.42) holds in this case as well. The fact that the set of all $\varkappa_m(\vec{\lambda})$ is finite, combined with the inequality (5.42), imply (5.40) for sufficiently small β . Using Lemma 5.5, as above we derive (5.27). Hence all terms in the expansion (5.23) are either zero or satisfy (5.27) or (5.28) implying consequently (5.22) and (5.18). \square

Here is the lemma used in the above proof.

Lemma 5.5. *Let assume that*

$$\begin{aligned} |\Psi_{i,\vartheta'} \Pi_{n',\zeta} \chi_{n,\zeta,\xi}^{(m)}(\mathbf{k}, \vec{k}) [\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}', \tau_1), \dots, \hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), \tau_1)]| &= 0 \\ &\text{for } (\mathbf{k}, \vec{k}) \in B_\beta \text{ and} \\ |\varphi_{n,\zeta,\xi}(\mathbf{k}, \vec{k})| &\geq \omega_* > 0 \text{ for } (\mathbf{k}, \vec{k}) \notin B_\beta, \text{ where } B_\beta \text{ as in (5.38)}. \end{aligned} \quad (5.46)$$

Then

$$\begin{aligned} \|\Psi(\cdot, \vartheta' \mathbf{k}_{*i}) \Pi_{n', \zeta} \mathcal{F}_{n, \zeta, \xi}^{(m)}(\tilde{\mathbf{w}}_{\lambda})\|_E &\leq \frac{4\varrho}{\omega_*} \|\chi^{(m)}\| \prod_j \|\hat{\mathbf{w}}_{\lambda_j}\|_E \\ &+ \frac{2\varrho\tau_*}{\omega_*} \|\chi^{(m)}\| \sum_i \|\partial_{\tau} \hat{\mathbf{w}}_{\lambda_i}\|_E \prod_{j \neq i} \|\hat{\mathbf{w}}_{\lambda_j}\|_E. \end{aligned} \quad (5.47)$$

PROOF. Notice that the oscillatory factor in (4.22) is equal to

$$\exp \left\{ i\varphi(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} = \frac{\varrho}{i\varphi(\mathbf{k}, \vec{k})} \partial_{\tau_1} \exp \left\{ i\varphi(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\}.$$

Denoting $\varphi_{n, \zeta, \xi} = \varphi$, $\Psi_{i, \vartheta'} \Pi_{n', \zeta} \chi_{n, \zeta, \xi}^{(m)} = \chi_{\vec{\eta}}^{(m)}$ and integrating (4.22) by parts with respect to τ_1 , we obtain

$$\begin{aligned} &\Psi(\mathbf{k}, \vartheta' \mathbf{k}_{*i}) \Pi_{n', \zeta} \mathcal{F}_{n, \zeta, \xi}^{(m)}(\tilde{\mathbf{w}}_{\lambda})(\mathbf{k}, \tau) \\ &= \int_B \Psi(\mathbf{k}, \vartheta' \mathbf{k}_{*i}) \frac{\varrho e^{i\varphi(\mathbf{k}, \vec{k}) \frac{\tau}{\varrho}}}{i\varphi(\mathbf{k}, \vec{k})} \chi_{\vec{\eta}}^{(m)}(\mathbf{k}, \vec{k}) \\ &\quad \times \hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}', \tau) \dots \hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), \tau) \tilde{d}^{(m-1)d} \vec{k} \\ &- \int_B \Psi(\mathbf{k}, \vartheta' \mathbf{k}_{*i}) \frac{\varrho}{i\varphi(\mathbf{k}, \vec{k})} \chi_{\vec{\eta}}^{(m)}(\mathbf{k}, \vec{k}) \\ &\quad \times \hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}', 0) \dots \hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}), 0) \tilde{d}^{(m-1)d} \vec{k} \\ &- \int_0^{\tau} \int_B \Psi(\mathbf{k}, \vartheta' \mathbf{k}_{*i}) \frac{\varrho e^{i\varphi(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho}}}{i\varphi(\mathbf{k}, \vec{k})} \chi_{\vec{\eta}}^{(m)}(\mathbf{k}, \vec{k}) \\ &\quad \times \partial_{\tau_1} [\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}') \dots \hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))] \tilde{d}^{(m-1)d} \vec{k} d\tau_1, \end{aligned} \quad (5.48)$$

where B is the set of $\mathbf{k}^{(i)}$ for which (5.38) holds. The relations (3.11) and (2.24) imply $|\chi_{\vec{\eta}}^{(m)}(\mathbf{k}, \vec{k})| \leq \|\chi^{(m)}\|$. Using then (5.46), the Leibnitz formula, (5.9) and (4.8), we obtain (5.47). \square

The main result of this subsection is the next theorem which, combined with Lemma 5.3, implies the wavepacket preservation, namely that the solution $\hat{\mathbf{u}}_{n, \vartheta}(\mathbf{k}, \tau)$ of (4.25) is a multi-wavepacket for all $\tau \in [0, \tau_*]$.

Theorem 5.6. *Assume that the conditions of Theorem 5.4 are fulfilled. Let $\hat{\mathbf{u}}_{n, \vartheta}(\mathbf{k}, \tau)$ for $n = n_l$, let $\hat{\mathbf{w}}_{l, \vartheta}(\mathbf{k}, \tau)$ be solutions to the respective*

systems (4.25) and (5.5), and let $\hat{\mathbf{w}}$ be defined by (5.13). Then for sufficiently small $\beta_0 > 0$

$$\|\hat{\mathbf{u}}_{n_l, \vartheta} - \Pi_{n_l, \vartheta} \hat{\mathbf{w}}\|_E \leq C\varrho + C'\beta^s, \quad 0 < \beta \leq \beta_0, \quad l = 1, \dots, N. \quad (5.49)$$

PROOF. Note that $\hat{\mathbf{u}}_{n_l, \vartheta} = \Pi_{n_l, \vartheta} \hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is a solution of (4.6) and, according to Theorem 4.7, $\|\hat{\mathbf{u}}\|_E \leq 2R$. Comparing Equations (4.6) and (5.17), which are $\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}$ and $\hat{\mathbf{w}} = \mathcal{F}(\hat{\mathbf{w}}) + \hat{\mathbf{h}} + \mathbf{D}(\hat{\mathbf{w}})$, we find that Lemma 4.6 can be applied. Then we notice that, by Lemma 4.5, \mathcal{F} has the Lipschitz constant $C_{F\tau_*}$ for such $\hat{\mathbf{u}}$. Taking $C_{F\tau_*} < 1$ as in Theorem 4.7, we obtain (5.49) from (4.17). \square

Notice that Theorem 2.9 is a direct corollary of Theorem 5.6 and Lemma 5.3.

An analogous assertion is proved in [7] for parameter-dependent equations of the form (2.1) with $\hat{\mathbf{F}}(\hat{\mathbf{U}}) = \hat{\mathbf{F}}(\hat{\mathbf{U}}, \varrho)$.

The following theorem shows that any multi-wavepacket solution to (4.6) yields a solution to the wavepacket interaction system (5.5).

Theorem 5.7. *Let $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ be a solution of (4.6). Assume that $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ and $\hat{\mathbf{h}}(\mathbf{k})$ are multi-wavepackets with nk -spectrum $S = \{(n_l, \mathbf{k}_{*l}), l = 1, \dots, N\}$ and regularity degree s . Let also $\Psi_{i_l, \vartheta} = \Psi_{i_l, \vartheta}$ be defined by (5.1). Then $\hat{\mathbf{w}}'_{l, \vartheta}(\mathbf{k}, \tau) = \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{\mathbf{u}}(\mathbf{k}, \tau)$ is a solution to the system (5.5) with $\hat{\mathbf{h}}(\mathbf{k})$ replaced by $\hat{\mathbf{h}}'(\mathbf{k}, \tau)$ satisfying*

$$\|\hat{\mathbf{h}}(\mathbf{k}) - \hat{\mathbf{h}}'(\mathbf{k}, \tau)\|_{L^1} \leq C\beta^s, \quad 0 \leq \tau \leq \tau_*, \quad (5.50)$$

and, if $\hat{\mathbf{w}}_{l, \vartheta}$ are solutions of (5.5) with original $\hat{\mathbf{h}}(\mathbf{k})$, then

$$\|\hat{\mathbf{w}}'_{l, \vartheta}(\mathbf{k}, \tau) - \hat{\mathbf{w}}_{l, \vartheta}\|_{L^1} \leq C\beta^s, \quad 0 \leq \tau \leq \tau_*. \quad (5.51)$$

PROOF. Multiplying (4.6) by $\Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta}$, we get

$$\begin{aligned} \hat{\mathbf{w}}'_{l, \vartheta} &= \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \mathcal{F}(\hat{\mathbf{u}})(\mathbf{k}, \tau) + \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \hat{\mathbf{h}}(\mathbf{k}), \\ \hat{\mathbf{w}}'_{l, \vartheta} &= \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \hat{\mathbf{u}}. \end{aligned} \quad (5.52)$$

Since $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ is a multi-wavepacket with regularity s , we have

$$\begin{aligned} \|\hat{\mathbf{u}}(\cdot, \tau) - \hat{\mathbf{w}}'(\cdot, \tau)\|_{L^1} &\leq C_\varepsilon \beta^s, \\ \text{where } \hat{\mathbf{w}}'(\cdot, \tau) &= \sum_{l, \vartheta} \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \hat{\mathbf{u}}(\cdot, \tau). \end{aligned} \quad (5.53)$$

Let us recast (5.52) in the form

$$\begin{aligned} \hat{\mathbf{w}}'_{l,\vartheta} &= \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \mathcal{F}(\hat{\mathbf{w}}')(\mathbf{k}, \tau) \\ &\quad + \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} [\hat{\mathbf{h}}(\mathbf{k}) + \hat{\mathbf{h}}''(\mathbf{k}, \tau)], \\ \hat{\mathbf{h}}''(\mathbf{k}, \tau) &= [\mathcal{F}(\hat{\mathbf{u}}) - \mathcal{F}(\hat{\mathbf{w}}')](\mathbf{k}, \tau). \end{aligned} \tag{5.54}$$

Denoting $\hat{\mathbf{h}}(\mathbf{k}) + \hat{\mathbf{h}}''(\mathbf{k}, \tau) = \hat{\mathbf{h}}'(\mathbf{k}, \tau)$, we observe that (5.54) has the form of (5.5) with $\hat{\mathbf{h}}(\mathbf{k})$ replaced by $\hat{\mathbf{h}}'(\mathbf{k}, \tau)$. The inequality (5.50) follows then from (5.53) and (4.13). Using Lemma 4.6, we obtain (5.51). \square

6. Reduction of Wavepacket Interaction System to an Averaged Interaction System

Our goal in this section is to substitute the wavepacket interaction system (5.5) with a simpler *averaged interaction system* which describes the evolution of wavepackets with the same accuracy, but has a simpler nonlinearity, and we follow here the approach developed in [7]. The reduction is a generalization of the classical averaging principle to the case of continuous spectrum, see [7] for a discussion and further simplification of the averaged interaction system. In the present paper, we do not need the further simplification to a minimal interaction system leading to a system of NLS-type equations which is done in [7].

6.1. Time averaged wavepacket interaction system.

Here we modify the wavepacket interaction system (5.5), substituting its nonlinearity with another one obtained by the time averaging, and prove that this substitution produces a small error of order ϱ . As the first step, we recast (5.5) in a slightly different form by using the expansions (5.15), (5.29) together with (5.33) and (5.34) and writing the nonlinearity in Equation (5.5) in the form

$$\begin{aligned} &\Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \mathcal{F}(\cdot, \tau) \\ &= \sum_{m \in \mathfrak{M}_F} \sum_{\vec{\lambda} \in \Lambda^m} \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \quad \vec{\lambda} = (\vec{l}, \vec{\zeta}), \end{aligned} \tag{6.1}$$

$$\mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k}, \tau) = \mathcal{F}_{n, \zeta, \vec{n}, \vec{\zeta}}^{(m)}[\hat{\mathbf{w}}_{\lambda_1} \dots \hat{\mathbf{w}}_{\lambda_m}](\mathbf{k}, \tau) \Big|_{\vec{n} = \vec{n}(\vec{l}), (n, \zeta) = (n_l, \vartheta)}, \tag{6.2}$$

with $\mathcal{F}_{n,\zeta,\vec{n},\vec{\zeta}}^{(m)}$ as in (4.22) and $\vec{n}(\vec{l})$ as in (5.31), and we call $\mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}})$ a *decorated monomial* $\mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}$ evaluated at $\vec{\mathbf{w}}_{\vec{\lambda}}$. Consequently, the wavepacket interaction system (5.5) can be written in the equivalent form

$$\hat{\mathbf{w}}_{l,\vartheta} = \sum_{m \in \mathfrak{M}_F} \sum_{\vec{\lambda} \in \Lambda^m} \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}}) + \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l,\vartheta} \hat{\mathbf{h}}, \tag{6.3}$$

$$l = 1, \dots, N, \quad \vartheta = \pm.$$

The construction of the above-mentioned time averaged equation reduces to discarding certain terms in the original system (6.3). First we introduce the following sets of indices related to the resonance equation (3.24) and Ω_m defined by (3.23):

$$\Lambda_{n_l,\vartheta}^m = \left\{ \vec{\lambda} = (\vec{l}, \vec{\zeta}) \in \Lambda^m : \Omega_m(\vartheta, n_l, \vec{\lambda}) = 0 \right\}, \tag{6.4}$$

and then the *time-averaged nonlinearity* \mathcal{F}_{av} by

$$\mathcal{F}_{av,n_l,\vartheta}(\vec{\mathbf{w}}) = \sum_{m \in \mathfrak{M}_F} \mathcal{F}_{n_l,\vartheta}^{(m)}, \quad \mathcal{F}_{n_l,\vartheta}^{(m)} = \sum_{\vec{\lambda} \in \Lambda_{n_l,\vartheta}^m} \mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}}), \tag{6.5}$$

where $\mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}$ are defined in (6.2).

Remark 6.1. Note that the nonlinearity $\mathcal{F}_{av,n_l,\vartheta}^{(m)}(\vec{\mathbf{w}})$ can be obtained from $\mathcal{F}_{n_l,\vartheta}^{(m)}$ by an averaging formula using an *averaging operator* A_T acting on polynomial functions $F : (\mathbb{C}^2)^N \rightarrow (\mathbb{C}^2)^N$ as follows:

$$(A_T F)_{j,\zeta} = \frac{1}{T} \int_0^T e^{-i\zeta\varphi_j t} \times F_{j,\zeta}(e^{i\varphi_1 t} u_{1,+}, e^{-i\varphi_1 t} u_{1,-}, \dots, e^{i\varphi_N t} u_{N,+}, e^{-i\varphi_N t} u_{N,-}) dt. \tag{6.6}$$

Using this averaging, we define for any polynomial nonlinearity $G : (\mathbb{C}^2)^N \rightarrow (\mathbb{C}^2)^N$ the averaged polynomial

$$G_{av,j,\zeta}(\vec{u}) = \lim_{T \rightarrow \infty} (A_T G)_{j,\zeta}(\vec{u}). \tag{6.7}$$

If the frequencies φ_j in (6.6) are generic, $G_{av,j,\zeta}(\vec{u})$ is always a universal nonlinearity. Note that $\mathcal{F}_{av,n_l,\vartheta}(\vec{\mathbf{w}})$ defined by (6.5) can be obtained by formula (6.7), where A_T is defined by formula (6.6) with frequencies $\varphi_j = \omega_{n_j}(\mathbf{k}_{*i_j})$ (it may be conditionally universal if the frequencies φ_j are subjected to a condition of the form (6.22), see the following subsection for details, in particular for definitions of universal and conditionally universal nonlinearities).

Finally, we introduce the wave interaction system with time-averaged nonlinearity as follows:

$$\hat{\mathbf{v}}_{l,\vartheta} = \Psi(\cdot, \vartheta \mathbf{k}_{*il}) \mathcal{F}_{\text{av},n_l,\vartheta}(\tilde{\mathbf{v}}) + \Psi(\cdot, \vartheta \mathbf{k}_{*il}) \Pi_{n_l,\vartheta} \hat{\mathbf{h}}, \quad l = 1, \dots, N, \vartheta = \pm. \quad (6.8)$$

Similarly to (5.7) we recast this system concisely as

$$\tilde{\mathbf{v}} = \mathcal{F}_{\text{av},\Psi}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \quad (6.9)$$

The following lemma is analogous to Lemmas 5.1, 4.5.

Lemma 6.2. *The operator $\mathcal{F}_{\text{av},\Psi}(\tilde{\mathbf{v}})$ is bounded for bounded $\tilde{\mathbf{v}} \in E^{2N}$, $\mathcal{F}_{\text{av},\Psi}(\mathbf{0}) = \mathbf{0}$. The polynomial operator $\mathcal{F}_{\text{av},\Psi}(\tilde{\mathbf{v}})$ satisfies the Lipschitz condition*

$$\|\mathcal{F}_{\text{av},\Psi}(\tilde{\mathbf{v}}_1) - \mathcal{F}_{\text{av},\Psi}(\tilde{\mathbf{v}}_2)\|_{E^{2N}} \leq C\tau_* \|\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2\|_{E^{2N}} \quad (6.10)$$

where C depends only on C_{χ} in (3.11), on the power of \mathcal{F} and on $\|\tilde{\mathbf{v}}_1\|_{E^{2N}} + \|\tilde{\mathbf{v}}_2\|_{E^{2N}}$, and, in particular, it does not depend on β, ϱ .

From Lemma 6.2 and the contraction principle we obtain the following theorem similar to Theorem 5.2.

Theorem 6.3. *Let $\|\tilde{\mathbf{h}}_{\Psi}\|_{E^{2N}} \leq R$. Then there exists $R_1 > 0$ and $\tau_* > 0$ such that Equation (6.9) has a solution $\tilde{\mathbf{v}} \in E^{2N}$ satisfying $\|\tilde{\mathbf{v}}\|_{E^{2N}} \leq R_1$, and such a solution is unique.*

The following theorem shows that the averaged interaction system introduced above provides a good approximation for the wave interaction system.

Theorem 6.4. *Let $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ be the solution of (6.8), and let $\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau)$ be the solution of (5.5). Then for sufficiently small β $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ is a wavepacket satisfying (5.10), (5.11) with $\hat{\mathbf{w}}$ replaced by $\hat{\mathbf{v}}$. In addition to that, there exists $\beta_0 > 0$ such that*

$$\begin{aligned} \|\hat{\mathbf{v}}_{l,\vartheta} - \hat{\mathbf{w}}_{l,\vartheta}\|_E &\leq C\varrho, \quad l = 1, \dots, N, \vartheta = \pm, \\ \text{for } 0 < \varrho &\leq 1, \quad 0 < \beta \leq \beta_0. \end{aligned} \quad (6.11)$$

PROOF. Formulas (5.10) and (5.11) for $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ follow from (6.8). We note that $\tilde{\mathbf{w}}$ is an approximate solution of (6.8), namely we have an estimate for $\mathbf{D}_{\text{av}}(\hat{\mathbf{w}}) = \hat{\mathbf{w}} - \mathcal{F}_{\text{av},\Psi} - \hat{\mathbf{h}}_{\Psi}$ which is similar to (5.17), (5.18):

$$\|\mathbf{D}_{\text{av}}(\hat{\mathbf{w}})\| = \|\hat{\mathbf{w}} - \mathcal{F}_{\text{av},\Psi} - \hat{\mathbf{h}}\|_{E^{2N}} \leq C\varrho \quad \text{if } 0 < \varrho \leq 1, \beta \leq \beta_0. \quad (6.12)$$

The proof of (6.12) is similar to the proof of (5.22) with minor simplifications thanks to the absence of terms with Ψ_{∞} . Using (6.12), we apply Lemma 4.6 and obtain (6.11). \square

6.1.1. Properties of averaged nonlinearities. In this section, we discuss elementary properties of nonlinearities obtained by formula (6.5). A key property of such nonlinearities $F_{j,\zeta}$ is the following homogeneity-like property:

$$\begin{aligned} F_{j,\zeta}(e^{i\varphi_1 t}u_{1,+}, e^{-i\varphi_1 t}u_{1,-}, \dots, e^{i\varphi_N t}u_{N,+}, e^{-i\varphi_N t}u_{N,-}) \\ = e^{i\zeta\varphi_j t}F_{j,\zeta}(u_{1,+}, u_{1,-}, \dots, u_{N,+}, u_{N,-}). \end{aligned} \tag{6.13}$$

The values of φ_i , $i = 1, \dots, N$, for which this formula holds depend on the resonance properties of the set S which enters (6.5) through the index set $\Lambda_{n_l, \vartheta}^m$. First, let us consider the simplest case, where φ_i are arbitrary. An example of such a nonlinearity is the function

$$F_{2,\zeta}(u_{1,+}, u_{1,-}, u_{2,+}, u_{2,-}) = u_{1,+}u_{1,-}u_{2,+}.$$

We call a nonlinearity which is obtained by formula (6.5) with a universal resonance invariant set S a *universal nonlinearity*.

Proposition 6.5. *If $F_{j,\zeta}$ is a universal nonlinearity, then (6.13) holds for arbitrary set of values φ_i , $i = 1, \dots, N$.*

PROOF. Note that the definition (6.5) of the averaged nonlinearity essentially is based on the selection of vectors $\vec{\lambda} = ((\zeta', l'), \dots, (\zeta^{(m)}, l_m)) \in \Lambda_{n_l, \vartheta}^m$ as in (6.4), which is equivalent to the resonance equation (3.24) with $n = n_l$, $\zeta = \vartheta$. This equation has the form

$$-\zeta\omega_n(\mathbf{k}_{**}) + \sum_{l=1}^N \delta_l \omega_l(\mathbf{k}_{*l}) = 0 \tag{6.14}$$

with

$$\mathbf{k}_{**} = -\zeta \sum_{l=1}^N \delta_l \mathbf{k}_{*l}, \tag{6.15}$$

where δ_l are the same as in (3.26). If $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m$ and

$$\begin{aligned} \vec{\mathbf{w}}_{\vec{\lambda}} &= (\hat{\mathbf{w}}_{\lambda_1} \dots \hat{\mathbf{w}}_{\lambda_m}) = (\hat{\mathbf{w}}_{\zeta', l_1} \dots \hat{\mathbf{w}}_{\zeta^{(m)}, l_m}) \\ &= (e^{-i\zeta' \varphi_{l_1}} \hat{\mathbf{v}}_{\zeta', l_1} \dots e^{-i\zeta^{(m)} \varphi_{l_m}} \hat{\mathbf{v}}_{\zeta^{(m)}, l_m}), \end{aligned}$$

then, using (6.2) and the multi-linearity of $\mathcal{F}^{(m)}$, we get

$$\mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}}) = e^{-i \sum \zeta^{(j)} \varphi_{l_j}} \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{v}}_{\vec{\lambda}})$$

and

$$\sum_{j=1}^m \zeta^{(j)} \varphi_{l_j} = \sum_{l=1}^N \delta_l \varphi_l, \tag{6.16}$$

where δ_l are the same as in (3.26). If we have a universal solution of (6.14), all coefficients at every $\omega_l(\mathbf{k}_{*l})$ cancel out ($\omega_n(\mathbf{k}_{**})$ also equals one of $\omega_l(\mathbf{k}_{*l})$, namely $\omega_n(\mathbf{k}_{**}) = \omega_{n_{I_0}}(\mathbf{k}_{*I_0})$). Using the notation (3.26), we see that a universal solution is determined by the system of equations on binary indices

$$\delta_l = \sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = 0, \quad l \neq I_0, \quad \delta_{I_0} = \sum_{j \in \vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta. \quad (6.17)$$

Obviously, the above condition does not involve values of ω_l and hence if δ_l, ζ correspond to a universal solution of (6.14), then we have the identity

$$-\zeta \varphi_{I_0} + \sum_{l=1}^N \delta_l \varphi_l = 0, \quad (6.18)$$

which holds for any $(\varphi_1, \dots, \varphi_N) \in \mathbb{C}^N$. □

Consider now the case, where the nk -spectrum S is resonance invariant, but may be not universal resonance invariant. Definition 3.8 of a resonance invariant nk -spectrum implies that the set $P(S)$ of all the solutions of (3.24) coincides with the set $P_{\text{int}}(S)$ of internal solutions. Hence all solutions of (6.14), (6.15) are internal, in particular $\mathbf{k}_{**} = \mathbf{k}_{*I_0}, \omega_n(\mathbf{k}_{**}) = \omega_{n_{I_0}}(\mathbf{k}_{*I_0})$ with some I_0 .

If we have a nonuniversal internal solution of (6.14), $\omega_l(\mathbf{k}_{*l})$ satisfy the following linear equation:

$$\zeta \omega_{n_{I_0}}(\mathbf{k}_{*I_0}) + \sum_{l=1}^N \delta_l \omega_l(\mathbf{k}_{*l}) = 0, \quad \zeta \mathbf{k}_{*I_0} + \sum_{l=1}^N \delta_l \mathbf{k}_{*l} = 0 \quad (6.19)$$

where at least one of b_j is nonzero. Note that if (6.19) is satisfied, we have additional (nonuniversal) solutions of (3.24) defined by

$$\sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = \delta_l, \quad l \neq I_0, \quad \sum_{j \in \vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta + \delta_{I_0}. \quad (6.20)$$

Now let us briefly discuss properties of Equations (6.20). The right-hand sides of the above system form a vector $\vec{b} = (b_1, \dots, b_N)$ with $b_l = \delta_l, l \neq I_0$, and $b_{I_0} = \zeta + \delta_{I_0}$. Note that $\vec{l} = (l_1, \dots, l_m)$ is uniquely defined by its level sets $\vec{l}^{-1}(l)$. For every l the number δ_{+l} of positive $\zeta^{(j)}$ and the number δ_{-l} of negative $\zeta^{(j)}$ with $j \in \vec{l}^{-1}(l)$ in (6.20) satisfy the equations

$$\delta_{+l} - \delta_{-l} = \delta_l, \quad \delta_{+l} + \delta_{-l} = |\vec{l}^{-1}(l)|, \quad (6.21)$$

where $|\vec{l}^{-1}(l)| = c_l$ is the cardinality (number of elements) of $\vec{l}^{-1}(l)$. Hence δ_{+l}, δ_{-l} are uniquely defined by $\delta_l, |\vec{l}^{-1}(l)|$. Hence the set of binary solutions $\vec{\zeta}$ of (6.20) with a given \vec{b} and a given $\vec{l} = (l_1, \dots, l_m)$ is determined by subsets of $\vec{l}^{-1}(l)$ with the cardinality δ_{+l} elements. Hence every solution with a given \vec{b} and a given \vec{l} can be obtained from one solution by permutations of indices j inside every level set $\vec{l}^{-1}(l)$. If \vec{b} is given and the cardinalities $|\vec{l}^{-1}(l)| = c_l$ are given, we can obtain different \vec{l} which satisfy (6.20) by choosing different decomposition of $\{1, \dots, m\}$ into subsets with given cardinalities c_l . For given \vec{b} and $\vec{c} = (c_1, \dots, c_m)$ we obtain this way the set (may be empty for some \vec{b}, \vec{c}) of all solutions of (6.20). Solutions with the same \vec{b} and \vec{c} we call equivalent.

When for a given wavepacket there are several nonequivalent nonuniversal solutions, the number of which is denoted by N_c , we obtain from (6.19) a system of equations with integer coefficients

$$\sum_{l=1}^N b_{l,i} \omega_l(\mathbf{k}_{*l}) = 0, \quad i = 1, \dots, N_c, \quad (6.22)$$

and solutions to (3.24) can be found from

$$\sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = b_{l,i}, \quad \text{for some } i, \quad 0 \leq i \leq N_c, \quad (6.23)$$

where to include universal solutions, we set $b_{l,0} = 0$.

Hence when a wavepacket is universally resonance invariant, we conclude that all terms in (6.5) satisfy (6.17). Since (6.18) holds, we get (6.13) for arbitrary $(\varphi_1, \dots, \varphi_N) \in \mathbb{C}^N$. If the wavepacket is conditionally universal with conditions (6.23), then, using (6.16) and (6.23), we conclude that (6.18) and (6.13) hold if $(\varphi_1, \dots, \varphi_N)$ satisfy the system of equations

$$\sum_{l=1}^N b_{l,i} \varphi_l = 0, \quad i = 1, \dots, N_c. \quad (6.24)$$

Now we would like to describe a special class of solutions of averaged equations. The evolution equation with an averaged nonlinearity has the form

$$\begin{aligned} \partial_\tau U_{j,+} &= \frac{-i}{\varrho} \mathcal{L}_j(-i\nabla) U_{j,+} + F_{j,+}(U_{1,+}, U_{1,-}, \dots, U_{N,+}, U_{N,-}), \\ \partial_\tau U_{j,-} &= \frac{i}{\varrho} \mathcal{L}_j(i\nabla) U_{j,+} + F_{j,-}(U_{1,+}, U_{1,-}, \dots, U_{N,+}, U_{N,-}), \\ & j = 1, \dots, N, \end{aligned} \quad (6.25)$$

where $\mathcal{L}(-i\nabla)$ is a linear scalar differential operator with constant coefficients. The characteristic property (6.13) implies that such a system admits special solutions of the form

$$U_{j,\zeta}(\mathbf{r}, \tau) = e^{-i\varphi_j\tau/\varrho} V_{j,\zeta}(\mathbf{r}), \quad (6.26)$$

where $V_{1,\zeta}(\mathbf{r})$ solve the time-independent *nonlinear eigenvalue problem*

$$\begin{aligned} -i\varphi_j V_{j,+} &= -i\mathcal{L}_j(-i\nabla)V_{j,+} + \varrho F_{j,+}(V_{1,+}, V_{1,-}, \dots, V_{N,+}, V_{N,-}), \\ i\varphi_j V_{j,-} &= i\mathcal{L}_j(i\nabla)V_{j,+} + \varrho F_{j,-}(V_{1,+}, V_{1,-}, \dots, V_{N,+}, V_{N,-}), \end{aligned} \quad (6.27)$$

$$j = 1, \dots, N.$$

6.1.2. Examples of universal and conditionally universal nonlinearities. Here we give a few examples of equations with averaged nonlinearities. When the multi-wavepacket is universal resonance invariant, the averaged wave interaction system involves NLS-type equations.

Example 6.6. The simplest example of (6.25) for one wavepacket ($N = 1$) and one spatial dimension ($d = 1$) is the nonlinear Schrödinger equation

$$\begin{aligned} \partial_\tau U_{1,+} &= -\frac{i}{\varrho} a_2 \partial_x^2 U_{1,+} - \frac{i}{\varrho} a_0 U_{1,+} + a_1 \partial_x U_{1,+} - iq U_{1,-} U_{1,+}^2, \\ \partial_\tau U_{1,-} &= \frac{i}{\varrho} a_2 \partial_x^2 U_{j,-} + \frac{i}{\varrho} a_0 U_{1,-} + a_1 \partial_x U_{1,-} + iq U_{1,+} U_{1,-}^2. \end{aligned} \quad (6.28)$$

Note that, by setting $y = x + a_1\tau/\varrho$, we can make $a_1 = 0$. Obviously, the nonlinearity

$$F_\zeta(U) = -i\zeta q U_{1,-\zeta} U_{1,\zeta}^2$$

satisfies (6.13):

$$i\zeta q e^{-i\zeta\varphi_1} U_{1,-\zeta} (e^{i\zeta\varphi_1} U_{1,\zeta})^2 = e^{i\zeta\varphi_1} i\zeta q U_{1,-\zeta} (U_{1,\zeta})^2.$$

The eigenvalue problem in this case takes the form

$$\begin{aligned} i\varphi_1 V_{1,+} &= -ia_2 \partial_x^2 V_{1,+} - ia_0 V_{1,+} + a_1 \partial_x V_{1,+} - iq V_{1,-} V_{1,+}^2, \\ -i\varphi_1 V_{1,-} &= ia_2 \partial_x^2 V_{j,-} + ia_0 V_{1,-} + a_1 \partial_x V_{1,-} + iq V_{1,+} V_{1,-}^2. \end{aligned} \quad (6.29)$$

If $a_1 = 0$ and we consider real-valued $V_{1,+} = V_{1,-}$, we obtain the equation

$$(\varphi_1 + a_0) V_{1,+} = -a_2 \partial_x^2 V_{1,+} - \varrho q V_{1,+}^3$$

or, equivalently,

$$\frac{(\varphi_1 + a_0)}{\varrho q} V_{1,+} + \frac{a_2}{\varrho q} \partial_x^2 V_{1,+} + V_{1,+}^3 = 0.$$

If

$$c^2 = \frac{a_2}{\varrho q} > 0, \quad \frac{(\varphi_1 + a_0)}{\varrho q} = -b^2 < 0, \quad (6.30)$$

the last equation takes the form

$$-b^2 V_{1,+} + c^2 \partial_x^2 V_{1,+} + V_{1,+}^3 = 0$$

with the family of classical soliton solutions

$$V_{1,+} = 2^{1/2} \frac{b}{\cosh(b(x - x_0)/c)}.$$

Note that the norm of the Fourier transform $\|\hat{V}_{1,+}\|_{L^1} = Cb$, where C is an absolute constant. Hence to have $\hat{V}_{1,+}$ bounded in L^1 uniformly in small ϱ according to (6.30), we should take $\varphi_1 = -a_0 - b^2 \varrho q$ with bounded b .

If the universal resonance invariant multi-wavepacket involves two wavepackets ($N = 2$) and the nonlinearity F is cubic, i.e., $\mathfrak{M}_F = \{3\}$, the semilinear system PDE with averaged nonlinearity has the form

$$\begin{aligned} \partial_t U_{2,+} &= -iL_2(i\nabla)U_{2,+} + U_{2,+}(Q_{2,1,+}U_{1,+}U_{1,-} + Q_{2,2,+}U_{2,+}U_{2,-}), \\ \partial_t U_{2,-} &= iL_2(-i\nabla)U_{2,-} + U_{2,-}(Q_{2,1,-}U_{1,+}U_{1,-} + Q_{2,2,-}U_{2,+}U_{2,-}), \\ \partial_t U_{1,+} &= -iL_1(i\nabla)U_{1,+} + U_{1,+}(Q_{1,1,+}U_{1,+}U_{1,-} + Q_{1,1,+}U_{2,+}U_{2,-}), \\ \partial_t U_{1,-} &= iL_1(-i\nabla)U_{1,-} + U_{1,-}(Q_{1,1,-}U_{1,+}U_{1,-} + Q_{1,1,-}U_{2,+}U_{2,-}). \end{aligned}$$

Obviously, (6.13) holds with arbitrary φ_1, φ_2 .

Now let us consider quadratic nonlinearities. In particular, let us consider the one-band symmetric case $\omega_n(\mathbf{k}) = \omega_1(\mathbf{k}) = \omega_1(-\mathbf{k})$, i.e., $J = 1$, $\mathfrak{M}_F = \{2\}$, and $m = 2$. Suppose that there is a multi-wavepacket involving two wavepackets with wavevectors $\mathbf{k}_{*1}, \mathbf{k}_{*2}$, i.e., $N = 2$. The resonance equation (3.24) takes now the form

$$-\zeta \omega_1(\zeta' \mathbf{k}_{*l_1} + \zeta'' \mathbf{k}_{*l_2}) + \zeta' \omega_1(\mathbf{k}_{*l_1}) + \zeta'' \omega_1(\mathbf{k}_{*l_2}) = 0, \quad (6.31)$$

where $l_1, l_2 \in \{1, 2\}$, $\zeta, \zeta', \zeta'' \in \{-1, 1\}$. All possible cases, and there are exactly four of them, correspond to the four well-known effects in the nonlinear optics: (i) $l_1 = l_2$, $\zeta' = \zeta''$ and $\zeta' = -\zeta''$ correspond respectively to second harmonic generation and nonlinear optical rectification; (ii) $l_1 \neq l_2$, $\zeta' = \zeta''$ and $\zeta' = -\zeta''$ correspond respectively to sum-frequency and difference-frequency interactions.

Let us suppose now that $\mathbf{k}_{*1}, \mathbf{k}_{*2} \neq 0$ and $\omega_1(\mathbf{k}_{*1}) \neq 0$, $\omega_1(\mathbf{k}_{*2}) \neq 0$, where the last conditions exclude the optical rectification, and that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}, \mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Consider first the case, where the wavepacket is universally resonance invariant.

Example 6.7. Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F , i.e., $\mathfrak{M}_F = \{2\}$. Let us pick two points \mathbf{k}_{*1} and $\mathbf{k}_{*2} \neq \pm\mathbf{k}_{*1}$ and assume that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}, \pm\mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Assume also that (i) $2\omega_1(\mathbf{k}_{*i}) \neq \omega_1(2\mathbf{k}_{*i}), i, j, l = 1, 2$, so there is no second harmonic generation; (ii) $\omega_1(\mathbf{k}_{*1}) \pm \omega_1(\mathbf{k}_{*2}) \neq \omega_1(\mathbf{k}_{*1} \pm \mathbf{k}_{*2})$, (no sum/difference-frequency interactions); (iii) $\omega_1(\mathbf{0}) \neq 0, \omega_j(\mathbf{k}_{*1}) \pm \omega_l(\mathbf{k}_{*2}) \neq 0$. Let set the nk -spectrum be the set $S_1 = \{(1, \mathbf{k}_{*1}), (1, \mathbf{k}_{*2})\}$. Then S_1 is resonance invariant.

In this case, (6.31) does not have solutions. Hence $\Lambda_{nl, \vartheta}^m = \emptyset$ and the averaged nonlinearity equals zero.

Now let us consider the case, where the wavepacket is not universal resonance invariant, but conditionally universal resonance invariant. In the following example, the conditionally resonance invariant spectrum allows for the second harmonic generation in the averaged system.

Example 6.8. Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F , i.e., $\mathfrak{M}_F = \{2\}$. Let us pick two points \mathbf{k}_{*1} and \mathbf{k}_{*2} such that $\mathbf{k}_{*2} = 2\mathbf{k}_{*1}$ and assume that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}, \pm\mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Assume also that (i) $2\omega_1(\mathbf{k}_{*1}) = \omega_1(2\mathbf{k}_{*1})$ (second harmonic generation); (ii) $\omega_i(\mathbf{k}_{*1}) \pm \omega_j(\mathbf{k}_{*2}) \neq \omega_l(\mathbf{k}_{*1} \pm \mathbf{k}_{*2}), i, j, l = 1, 2$ (no sum-/difference-frequencies interaction); (iii) $\omega_1(\mathbf{0}) \neq 0, \omega_j(\mathbf{k}_{*1}) \pm \omega_l(\mathbf{k}_{*2}) \neq 0$. Let set the nk -spectrum be the set $S = \{(1, \mathbf{k}_{*1}), (1, \mathbf{k}_{*2})\}$. Then S is resonance invariant. The condition (6.19) is takes here the form

$$2\omega_1(\mathbf{k}_{*1}) - \omega_1(\mathbf{k}_{*2}) = 0, \quad 2\mathbf{k}_{*1} - \mathbf{k}_{*2} = 0,$$

and the condition (6.24) turns into

$$2\omega_1(\mathbf{k}_{*1}) - \omega_1(\mathbf{k}_{*2}) = 0.$$

The wavepacket interaction system for such a multi-wavepacket has the form

$$\begin{aligned} \partial_t U_{2,+} &= -iL_2(i\nabla)U_{2,+} + Q_{2,2,+}U_{1,+}U_{1,+}, \\ \partial_t U_{2,-} &= iL_2(-i\nabla)U_{2,-} + Q_{2,2,-}U_{1,-}U_{1,-}, \\ \partial_t U_{1,+} &= -iL_1(i\nabla)U_{1,+} + Q_{1,2,+}U_{2,+}U_{1,-}, \\ \partial_t U_{1,-} &= iL_1(-i\nabla)U_{1,-} + Q_{1,2,-}U_{2,-}U_{1,+}. \end{aligned}$$

6.2. Invariance of multi-particle wavepackets.

The following lemma shows that particle wavepackets are preserved under action of certain types of nonlinearities with elementary susceptibilities as in (4.20). In the following section, we show, in particular, that universal nonlinearities are composed of such terms.

Lemma 6.9. *Let the components $\hat{\mathbf{w}}_{l_i, \zeta} = \hat{\mathbf{w}}_{\lambda_i}$ of $\tilde{\mathbf{w}}_{\vec{\lambda}} = \hat{\mathbf{w}}_{\lambda_1} \dots \hat{\mathbf{w}}_{\lambda_m}$ be particle-like wavepackets in the sense of Definition 2.2, and let $\mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ be as in (6.5). Assume that*

$$\hat{\mathbf{w}}_{l_i, \zeta}(\mathbf{k}, \beta) = 0 \text{ if } |\mathbf{k} - \zeta \mathbf{k}_{*l_i}| \geq \beta^{1-\varepsilon}, \quad \zeta = \pm, \quad i = 1, \dots, m. \quad (6.32)$$

Assume that the vector index $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m$ is such a vector which has at least one component $\lambda_j = (\zeta_j, l_j)$ such that

$$\nabla \omega_{n_l}(\mathbf{k}_{*l}) = \nabla \omega_{n_{l_j}}(\mathbf{k}_{*l_j}). \quad (6.33)$$

Then for any $\mathbf{r}_* \in \mathbb{R}^d$

$$\begin{aligned} & \|\nabla_{\mathbf{k}}(e^{-i\mathbf{r}_* \cdot \mathbf{k}} \Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon}) \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}))\|_E \\ & \leq C\tau_* \|\nabla_{\mathbf{k}} e^{-i\mathbf{r}_* \cdot \mathbf{k}^{(j)}} \mathbf{w}_{l_j}\|_E \prod_{i \neq j} \|\mathbf{w}_{l_i, \zeta_i}\|_E \\ & + C\tau_* \left(\beta^{-1+\varepsilon} + \frac{\beta^{1-\varepsilon}}{\varrho} \right) \prod_{j=1}^m \|\mathbf{w}_{l_j, \zeta_j}\|_E, \end{aligned} \quad (6.34)$$

where C does not depend on \mathbf{r}_* and small β, ϱ .

PROOF. Note that

$$\mathbf{r}_* \cdot \mathbf{k} = \mathbf{r}_* \cdot (\mathbf{k}' + \dots + \mathbf{k}^{(m)}).$$

We have by (4.22)

$$\begin{aligned} \nabla_{\mathbf{k}} e^{-i\mathbf{r}_* \cdot \mathbf{k}} \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k}, \tau) &= \nabla_{\mathbf{k}} \int_0^\tau \int_{[-\pi, \pi]^{2d}} \exp \left\{ i\varphi_{\vartheta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} \\ & \times \Psi e^{-i\mathbf{r}_* \cdot \mathbf{k}} \chi_{\vartheta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) \mathbf{w}_{l_1, \zeta'}(\mathbf{k}') \dots \mathbf{w}_{l_m, \zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \tilde{\mathbf{d}}^{(m-1)d} \vec{k} d\tau_1. \end{aligned} \quad (6.35)$$

Without loss of generality, we assume that in (6.33) $l_j = l_m$ (the general case is reduced to this one by a re-enumeration of variables of integration). By the Leibnitz formula,

$$\nabla_{\mathbf{k}} [\Psi e^{-i\mathbf{r}_* \cdot \mathbf{k}} \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})](\mathbf{k}, \tau) = I_1 + I_2 + I_3, \quad (6.36)$$

where

$$I_1 = \int_0^\tau \int_{[-\pi, \pi]^{(m-1)d}} \nabla_{\mathbf{k}} \exp \left\{ i\varphi_{\theta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} - i\mathbf{r}_* \mathbf{k} \right\} \Psi \chi_{\theta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) e^{-i\mathbf{r}_* \mathbf{k}'} \\ \times \mathbf{w}_{l_1, \zeta'}(\mathbf{k}') \dots e^{-i\mathbf{r}_* \mathbf{k}^{(m)}} \mathbf{w}_{l_m, \zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \tilde{d}^{(m-1)d} \vec{k} d\tau_1,$$

$$I_2 = \int_0^\tau \int_{[-\pi, \pi]^{(m-1)d}} \Psi \exp \left\{ i\varphi_{\theta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} - i\mathbf{r}_* \mathbf{k} \right\} \\ \times [\nabla_{\mathbf{k}} (\Psi(\mathbf{k}, \mathbf{k}_{*l}, \beta^{1-\varepsilon}) \chi_{\theta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}))] e^{-i\mathbf{r}_* \mathbf{k}'} \mathbf{w}_{l_1, \zeta'}(\mathbf{k}') \dots e^{-i\mathbf{r}_* \mathbf{k}^{(m)}} \\ \times \mathbf{w}_{l_m, \zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \tilde{d}^{(m-1)d} \vec{k} d\tau_1,$$

$$I_3 = \int_0^\tau \int_{[-\pi, \pi]^{(m-1)d}} \exp \left\{ i\varphi_{\theta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} - i\mathbf{r}_* \mathbf{k} \right\} \Psi \chi_{\theta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) e^{-i\mathbf{r}_* \mathbf{k}'} \\ \times \mathbf{w}_{l_1, \zeta'}(\mathbf{k}') \dots \nabla_{\mathbf{k}} (e^{-i\mathbf{r}_* \mathbf{k}^{(m)}(\mathbf{k}, \vec{k})} \mathbf{w}_{l_m, \zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))) \tilde{d}^{(m-1)d} \vec{k} d\tau_1.$$

Since $\mathbf{w}_{j, \zeta}$ are bounded, we have

$$\|e^{-i\mathbf{r}_* \mathbf{k}^{(j)}} \mathbf{w}_{l_j, \zeta^{(j)}}(\mathbf{k}^{(j)})\|_{L^1} \leq \|\mathbf{w}_{l_j, \zeta^{(j)}}(\mathbf{k}^{(j)})\|_{L^1} \leq C_1, \quad j = 1, \dots, m. \quad (6.37)$$

Using (4.8) and (6.37), we get

$$|I_3| \leq \|\chi^{(m)}\| \prod_{j=1}^{m-1} \|\mathbf{w}_{l_j, \zeta^{(j)}}\|_E \int_0^\tau \|\nabla_{\mathbf{k}} e^{-i\mathbf{r}_* \mathbf{k}^{(m)}(\mathbf{k}, \vec{k})} \mathbf{w}_{l_m, \zeta^{(m)}}\|_E d\tau_1. \quad (6.38)$$

From (6.37), (2.25), (3.13) and the smoothness of $\Psi(\mathbf{k}, \mathbf{k}_{*l}, \beta^{1-\varepsilon})$ we get

$$|I_2| \leq C_2 \beta^{-1+\varepsilon} \prod_{j=1}^m \|\mathbf{w}_{l_j, \zeta_j}\|_E. \quad (6.39)$$

Now let us estimate I_1 . Using (4.23), we obtain

$$I_1 = \int_0^\tau \int_{[-\pi, \pi]^{(m-1)d}} \left[\exp \left\{ i\varphi_{\theta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} \right] \\ \times \frac{\tau_1}{\varrho} [-\theta \nabla_{\mathbf{k}} \omega_{n_l}(\mathbf{k}) + \zeta^{(m)} \nabla_{\mathbf{k}} \omega_{n_{l_m}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))] \\ \times \chi_{\theta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) \mathbf{w}_{l_1, \zeta'}(\mathbf{k}') \dots \mathbf{w}_{l_m, \zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k})) \tilde{d}^{(m-1)d} \vec{k} d\tau_1. \quad (6.40)$$

The difficulty in the estimation of the integral I_1 comes from the factor τ_1/ϱ since ϱ is small. Since (6.32) holds, it is sufficient to estimate I_1 if

$$|\mathbf{k}^{(j)} - \zeta^{(j)}\mathbf{k}_{*n_j}| \leq \beta^{1-\varepsilon} \text{ for all } j. \tag{6.41}$$

According to (3.18), since $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m$, we have

$$\mathbf{k}^{(m)}(\mathbf{k}_{*n_l}, \vec{k}_*) = \mathbf{k}_{*n_{l_m}}.$$

Hence, using (6.33) and (4.23), we obtain

$$\nabla_{\mathbf{k}} \varphi_{\theta, \vec{\zeta}}(\mathbf{k}_{*n_l}, \vec{k}_*) = [-\theta \nabla_{\mathbf{k}} \omega_{n_l}(\mathbf{k}_{*n_l}) + \zeta^{(m)} \nabla_{\mathbf{k}} \omega_{n_{l_m}}(\mathbf{k}^{(m)}(\mathbf{k}_{*n_l}, \vec{k}_*))] = 0. \tag{6.42}$$

Using (3.2), we conclude that, in a vicinity of \vec{k}_* defined by (6.41), we have

$$t|[-\theta \nabla_{\mathbf{k}} \omega(\mathbf{k}) + \zeta^{(m)} \nabla_{\mathbf{k}} \omega(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))]| \leq 2(m+1)C_{\omega, 2}\beta^{1-\varepsilon}.$$

This yields the estimate

$$|I_1| \leq C_3\beta^{1-\varepsilon}/\varrho. \tag{6.43}$$

Combining (6.43), (6.39) and (6.38), we obtain (6.48). \square

We introduce a β -dependent Banach space E^1 of differentiable functions of variable \mathbf{k} by the formula

$$\|\mathbf{w}\|_{E^1(\mathbf{r}_*)} = \beta^{1+\varepsilon} \|\nabla_{\mathbf{k}}(e^{-i\mathbf{r}_*\mathbf{k}}\mathbf{w})\|_E + \|\mathbf{w}\|_E. \tag{6.44}$$

We use for $2N$ -component vectors with elements $\mathbf{w}_i(\mathbf{k}) \in E^2$ the following notation:

$$\begin{aligned} \tilde{\mathbf{w}}(\mathbf{k}) &= (\mathbf{w}_1(\mathbf{k}), \dots, \mathbf{w}_N(\mathbf{k})), \\ \tilde{\mathbf{r}}_* &= (\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N}), \quad \mathbf{w}_i(\mathbf{k}) = (\mathbf{w}_{i,+}(\mathbf{k}), \mathbf{w}_{i,-}(\mathbf{k})), \\ e^{-i\tilde{\mathbf{r}}_*\mathbf{k}}\tilde{\mathbf{w}}(\mathbf{k}) &= (e^{-i\mathbf{r}_{*1}\mathbf{k}}\mathbf{w}_1(\mathbf{k}), \dots, e^{-i\mathbf{r}_{*N}\mathbf{k}}\mathbf{w}_N(\mathbf{k})), \end{aligned} \tag{6.45}$$

Similarly to (5.6) we introduce the space $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ with the norm

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} = \sum_{l, \vartheta} \|\hat{\mathbf{w}}_{l, \vartheta}\|_{E^1(\mathbf{r}_{*l})}. \tag{6.46}$$

The following proposition is obtained by comparing (6.44) and (2.33).

Proposition 6.10. *A multi-wavepacket $\tilde{\mathbf{w}}$ is a multi-particle one with positions $\mathbf{r}_{*1}, \dots, \mathbf{r}_{*N}$ if and only if*

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq C,$$

where the constant C does not depend on β , $0 < \beta \leq 1/2$, and $\tilde{\mathbf{r}}_*$.

In view of the above, we call $E^1(\mathbf{r}_*)$ and $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ particle spaces. We also use the notation

$$\begin{aligned} \Psi_2 \tilde{\mathbf{w}}_{\vec{\lambda}} &= (\Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon}/2) \mathbf{w}_{\lambda_1}, \dots, \Psi(\cdot, \mathbf{k}_{*l_m}, \beta^{1-\varepsilon}/2) \mathbf{w}_{\lambda_m}), \\ \mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{w}}^m) &= \Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon}) \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\Psi_2 \tilde{\mathbf{w}}_{\vec{\lambda}}). \end{aligned}$$

Lemma 6.11. *Let $\tilde{\mathbf{w}}, \tilde{\mathbf{v}} \in (E^1)^{2N}(\tilde{\mathbf{r}}_*)$ and $\mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ be as in (6.5). Assume that the vector index $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m$ is such a vector which has at least one component $\lambda_j = (\zeta_j, l_j)$ with $l_j = l$. Assume that (1.9) holds and $\Psi(\cdot, \mathbf{k}_*, \beta^{1-\varepsilon})$ is defined in (2.25). Let $\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq 2R$. Then*

$$\|\mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{w}})\|_{E^1(\mathbf{r}_{*l})} \leq C\tau_* \|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)}^{m-1} \|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \quad (6.47)$$

where C does not depend on β , $0 < \beta \leq 1/2$, and on $\tilde{\mathbf{r}}_*$, \mathbf{r}_{*l} and $\tilde{\mathbf{r}}_*$ is defined by (6.45). If $\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq 2R$ the following Lipschitz inequality holds:

$$\|\mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{w}}) - \mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{v}})\|_{E^1(\mathbf{r}_{*l})} \leq C\tau_* \|\tilde{\mathbf{w}} - \tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)}. \quad (6.48)$$

where C does not depend on β , $0 < \beta \leq 1/2$, and on $\tilde{\mathbf{r}}_*$, \mathbf{r}_{*l} .

PROOF. Note that $\Psi_2 \tilde{\mathbf{w}}_{\vec{\lambda}}$ and $\Psi_2 \tilde{\mathbf{v}}_{\vec{\lambda}}$ are wavepackets in the sense of Definition 2.2. To obtain (6.47), we apply the inequality (6.34) and use (1.9); for the part of the E^1 -norm without \mathbf{k} -derivatives we use (4.10). Using multilinearity of $\mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}$, we observe that

$$\begin{aligned} &\mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{w}}) - \mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{v}}) \\ &= \sum_{j=1}^m \mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi_2}^{(m)}(\mathbf{w}_{\lambda_1}, \dots, \mathbf{w}_{\lambda_j} - \mathbf{v}_{\lambda_j}, \mathbf{v}_{\lambda_{j+1}}, \dots, \mathbf{v}_{\lambda_m}). \end{aligned} \quad (6.49)$$

We can apply to every term the inequality (6.34). Multiplying (6.34) by $\beta^{1+\varepsilon}$ and using (1.9), we deduce (6.48). \square

Now we consider a system similar to (6.8),

$$\hat{\mathbf{v}}_{l, \vartheta} = \mathcal{F}_{\text{av}, \Psi_2, n_l, \vartheta}(\tilde{\mathbf{v}}) + \Psi(\cdot, \vartheta \mathbf{k}_{*i_l}) \Pi_{n_l, \vartheta} \hat{\mathbf{h}}, \quad l = 1, \dots, N, \vartheta = \pm, \quad (6.50)$$

where $\mathcal{F}_{\text{av}, \Psi, n_l, \vartheta}$ is defined by a formula similar to(6.5):

$$\mathcal{F}_{\text{av}, \Psi_2, n_l, \vartheta}(\tilde{\mathbf{v}}) = \sum_{m \in \mathfrak{M}_F} \mathcal{F}_{n_l, \vartheta}^{(m)}, \quad \mathcal{F}_{n_l, \vartheta}^{(m)} = \sum_{\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m} \mathcal{F}_{n_l, \vartheta, \vec{\lambda}, \Psi}^{(m)}(\tilde{\mathbf{v}}). \quad (6.51)$$

The system (6.50) can be written in the form similar to (6.9)

$$\tilde{\mathbf{v}} = \mathcal{F}_{\text{av}, \Psi_2}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \tag{6.52}$$

Theorem 6.12 (solvability in particle spaces). *Let the initial data $\tilde{\mathbf{h}}$ in the averaged wavepacket interaction system (6.52) be a multi-particle wavepacket $\hat{\mathbf{h}}(\beta, \mathbf{k})$ with the nk -spectrum S as in (2.39), regularity degree s , and positions \mathbf{r}_{*l} , $l = 1, \dots, N$. Let $\|\tilde{\mathbf{h}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq R$. Assume that S is universally resonance invariant in the sense of Definition 3.8. Then there exists $\tau_{**} > 0$ which does not depend on $\tilde{\mathbf{r}}_*$, β , and ϱ such that if $\tau_* \leq \tau_{**}$, Equation (6.52) has a unique solution $\tilde{\mathbf{v}}$ in $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ such that*

$$\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq 2R, \tag{6.53}$$

where R does not depend on ϱ, β , and $\tilde{\mathbf{r}}_*$. This solution is a multi-particle wavepacket with positions \mathbf{r}_{*l} .

PROOF. Since S is universally resonance invariant, every vector index $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m$ has at least one component $\lambda_j = (\zeta_j, l_j)$ with $l_j = l$. Hence Lemma 6.11 is applicable and, according to (6.48), the operator $\mathcal{F}_{\text{av}, \Psi_2}$ defined by (6.51) is Lipschitz in the ball $\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq 2R$ with a Lipschitz constant $C'\tau_*$, where C' does not depend on ϱ, β , and $\tilde{\mathbf{r}}_*$. We choose τ_{**} so that $C'\tau_{**} \leq 1/2$ and use Lemma 4.6. According to this lemma, Equation (6.52) has a solution $\tilde{\mathbf{v}}$ which satisfies (6.53). This solution is a multi-particle wavepacket according to Proposition 6.10. \square

Theorem 6.13 (particle wavepacket approximation). *Let the initial data $\hat{\mathbf{h}}$ in the integral equation (2.14) with solution $\hat{\mathbf{u}}(\tau, \beta; \mathbf{k})$ be an multi-particle wavepacket $\hat{\mathbf{h}}(\beta, \mathbf{k})$ with the nk -spectrum S as in (2.39), regularity degree s , and positions \mathbf{r}_{*l} $l = 1, \dots, N$, and let the components of $\hat{\mathbf{h}}(\beta, \mathbf{k})$ satisfy the inequality $\|\tilde{\mathbf{h}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leq R$. Let $\tau_* \leq \tau_{**}$. Assume that S is universally resonance invariant in the sense of Definition 3.8. We define $\hat{\mathbf{v}}(\tau, \beta; \mathbf{k})$ by the formula*

$$\hat{\mathbf{v}}(\tau, \beta; \mathbf{k}) = \sum_{l=1}^N \sum_{\zeta=\pm} \hat{\mathbf{v}}_{l, \vartheta}(\tau, \beta; \mathbf{k}), \quad l = 1, \dots, N, \tag{6.54}$$

where $\hat{\mathbf{v}}_{l, \vartheta}(\tau, \beta; \mathbf{k})$ is a solution of (6.8). Then every such $\hat{\mathbf{v}}_l(\mathbf{k}; \tau, \beta)$ is a particle-like wavepacket with position \mathbf{r}_{*l} and

$$\sup_{0 \leq \tau \leq \tau_*} \|\hat{\mathbf{u}}(\tau, \beta; \mathbf{k}) - \hat{\mathbf{v}}(\tau, \beta; \mathbf{k})\|_{L^1} \leq C_1 \varrho + C_2 \beta^s, \tag{6.55}$$

where the constant C_1 does not depend on ϱ, s , and β and the constant C_2 does not depend on ϱ, β .

PROOF. Let $\tilde{\mathbf{v}} \in (E^1)^{2N}(\tilde{\mathbf{r}}_*)$ be the solution of Equation (6.52) which exists by Theorem 6.12. It is a particle-like wavepacket. Note that

$$\Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon}/2)\Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon}) = \Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon})$$

and a solution of (6.52) has the form $\hat{\mathbf{v}}_{l,\vartheta}(\tau, \beta; \mathbf{k}) = \Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon})[\dots]$. Consequently, for such solutions $\Psi_2 \tilde{\mathbf{v}}_{\tilde{\lambda}} = \tilde{\mathbf{v}}_{\tilde{\lambda}}$ the nonlinearity $\mathcal{F}_{n_l, \vartheta, \tilde{\lambda}, \Psi_2}^{(m)}(\tilde{\mathbf{v}})$ coincides with the equation $\Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon})\mathcal{F}_{n_l, \vartheta, \tilde{\xi}(\tilde{\lambda})}^{(m)}(\tilde{\mathbf{v}}_{\tilde{\lambda}})$ and Equation (6.50) coincides with (6.8). Hence $\tilde{\mathbf{v}}$ is a solution of (6.8). The estimate (6.55) follows from the estimates (6.11) and (5.49). \square

Now, we are able to prove Theorem 2.10.

Corollary 6.14 (proof of Theorem 2.10). *If the conditions of Theorem 2.10 are satisfied, the statement of Theorem 2.10 holds.*

PROOF. Note that the functions $\hat{\mathbf{w}}'_{l,\vartheta}(\mathbf{k}, \tau) = \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{\mathbf{u}}(\mathbf{k}, \tau)$, $\theta = \pm$, in Theorem 5.7 are two components of $\hat{\mathbf{u}}_l(\tau, \beta; \mathbf{k})$ in (2.45). Hence (5.51) implies that

$$\|\hat{\mathbf{u}}_l - \hat{\mathbf{w}}_{l,+} - \hat{\mathbf{w}}_{l,-}\|_E \leq C'\beta^s, \quad 0 < \beta \leq \beta_0, \quad (6.56)$$

where $\hat{\mathbf{w}}_{l,\vartheta}$ are solutions to (5.5). According to (6.11), if $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ is the solution of (6.8), we have

$$\|\hat{\mathbf{v}}_{l,\vartheta} - \hat{\mathbf{w}}_{l,\vartheta}\|_E \leq C_\varrho, \quad l = 1, \dots, N; \quad \vartheta = \pm. \quad (6.57)$$

Hence

$$\|\hat{\mathbf{u}}_l - \hat{\mathbf{v}}_{l,+} - \hat{\mathbf{v}}_{l,-}\|_E \leq C_\varrho + C'\beta^s, \quad 0 < \beta \leq \beta_0. \quad (6.58)$$

This inequality implies (2.46). We have proved that $\hat{\mathbf{v}}_{l,\vartheta}$ is a particle-like wavepacket as in Theorem 6.13. The estimate (6.58) implies that $\hat{\mathbf{u}}_l$ is equivalent to $\hat{\mathbf{v}}_l = \hat{\mathbf{v}}_{l,+} + \hat{\mathbf{v}}_{l,-}$ in the sense of (2.42) of degree $s_1 = \min(s, s_0)$. \square

7. Superposition Principle and Decoupling of the Wavepacket Interaction System

In this section, we give the proof of the superposition principle of [8] which is based on the study of the wavepacket interaction system (6.8). We

show that when we omit cross-terms in the averaged wavepacket interaction system, the resulting error is estimated by $\frac{\rho}{\beta^{1+\varepsilon}} |\ln \beta|$, i.e., component wavepackets evolve essentially independently and the time averaged wavepacket interaction system almost decouples.

Let $\mathcal{F}_{\text{av},n_l,\vartheta}$ be defined by (6.5), and let a decoupled nonlinearity $\mathcal{F}_{\text{av},n_l,\vartheta,\text{diag}}$ be defined by

$$\begin{aligned} \mathcal{F}_{\text{av},n_l,\vartheta,\text{diag}}(\tilde{\mathbf{w}}) &= \sum_{m \in \mathfrak{M}_F} \mathcal{F}_{n_l,\vartheta}^{(m)} \mathcal{F}_{n_l,\vartheta,\text{diag}}^{(m)}(\tilde{\mathbf{w}}) \\ &= \sum_{\vec{\lambda} \in \Lambda_{n_l,\vartheta}^{m,\text{diag}}} \mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \end{aligned} \quad (7.1)$$

where the set of indices $\Lambda_{n_l,\vartheta}^{m,\text{diag}}$ is defined by the formula

$$\Lambda_{n_l,\vartheta}^{m,\text{diag}} = \{ \vec{\lambda} = (\vec{l}, \vec{\zeta}) \in \Lambda_{n_l,\vartheta}^m : l_j = l, j = 1, \dots, m \}. \quad (7.2)$$

Note that $\mathcal{F}_{n_l,\vartheta,\text{diag}}^{(m)}$ in (7.1) depends only on $\mathbf{w}_{l,+}$ and $\mathbf{w}_{l,-}$:

$$\mathcal{F}_{n_l,\vartheta,\text{diag}}^{(m)}(\tilde{\mathbf{w}}) = \mathcal{F}_{\vartheta,\text{diag},l}^{(m)}(\mathbf{w}_l), \quad \mathbf{w}_l = (\mathbf{w}_{l,+}, \mathbf{w}_{l,-}). \quad (7.3)$$

The coupling between different variables \mathbf{v}_l in (6.8) is caused by nondiagonal terms

$$\mathcal{F}_{\text{av},n_l,\vartheta,\text{coup}}(\tilde{\mathbf{w}}) = \mathcal{F}_{\text{av},n_l,\vartheta}(\tilde{\mathbf{w}}) - \mathcal{F}_{\text{av},n_l,\vartheta,\text{diag}}(\tilde{\mathbf{w}}). \quad (7.4)$$

Obviously, Equation (6.9) can be written in the form

$$\tilde{\mathbf{v}} = \mathcal{F}_{\text{av},\Psi,\text{diag}}(\tilde{\mathbf{v}}) + \mathcal{F}_{\text{av},\Psi,\text{coup}}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \quad (7.5)$$

The system of decoupled equations has the form

$$\tilde{\mathbf{v}}^{\text{diag}} = \mathcal{F}_{\text{av},\Psi,\text{diag}}(\tilde{\mathbf{v}}^{\text{diag}}) + \tilde{\mathbf{h}}_{\Psi} \quad (7.6)$$

or, when written in components,

$$\mathbf{v}^{\text{diag},l} = \mathcal{F}_{\text{av},\Psi,\text{diag},l}^{(m)}(\mathbf{v}^{\text{diag},l}) + \mathbf{h}_{\Psi,l}, \quad l = 1, \dots, N. \quad (7.7)$$

We prove that the contribution of $\mathcal{F}_{\text{av},\Psi,\text{coup}}$ in (7.5) is small. The proof is based on the following lemma.

Lemma 7.1 (small coupling terms). *Let $\mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ be as in (6.5), let all the components \mathbf{w}_{λ_i} of $\tilde{\mathbf{w}}_{\vec{\lambda}}$ satisfy (6.32) and be wavepackets in the sense of Definition 2.1, and let (1.9) hold. Assume also that: (i) the vector index $\vec{\lambda}$ has at least two components $\lambda_i = (\zeta_i, l_i)$ and $\lambda_j = (\zeta_j, l_j)$ with*

$l_i \neq l_j$; (ii) both \mathbf{w}_{λ_i} and \mathbf{w}_{λ_j} are particle wavepackets in the sense of Definition 2.2; (iii) either (2.51) or (2.54) holds. Then for small β and ϱ

$$\|\mathcal{F}_{n_i, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}})\|_{E^N} \leq C \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|. \quad (7.8)$$

PROOF. Since \mathbf{k}_{*l} are not band-crossing points, according to Definition 3.1 and Condition 3.2 the inequalities (3.2) and (3.13) hold. According to the assumption of the theorem, at least two $\hat{\mathbf{w}}_{l_j}$ are different for different j . Let us assume that $l_{j_1} = l_1, l_{j_2} = l_m, l_1 \neq l_m$ (the general case can be easily reduced to this one by relabeling variables). Since $\hat{\mathbf{w}}_{l_1}$ and $\hat{\mathbf{w}}_{l_m}$ are particle wavepackets, they satisfy (2.33) with \mathbf{r} replaced by \mathbf{r}_{l_1} and \mathbf{r}_{l_m} respectively. Let us rewrite the integral with respect to τ_1 in (4.22) as

$$\mathcal{F}_{n_i, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k}, \tau) = \int_0^\tau \int_{\mathbb{D}_m} \exp \left\{ i\varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} \right\} A_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) \tilde{d}^{(m-1)d} \vec{k} d\tau_1, \quad (7.9)$$

where

$$A_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) = \chi_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}) \mathbf{w}_{l_1}(\mathbf{k}') \dots \mathbf{w}_{l_m}(\mathbf{k}^{(m)}), \quad (7.10)$$

and then rewrite (7.9) in the form

$$\begin{aligned} \mathcal{F}_{n_i, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\vec{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k}, \tau) &= \mathcal{F}_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{w}_{l_1} \dots \mathbf{w}_{l_m})(\mathbf{k}, \tau) \\ &= \int_0^\tau \int_{\mathbb{D}_m} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) \tilde{d}^{(m-1)d} \vec{k} d\tau_1, \end{aligned} \quad (7.11)$$

where

$$\begin{aligned} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) &= \exp \left\{ i\varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \frac{\tau_1}{\varrho} - i\mathbf{r}_{l_1} \mathbf{k}' - i\mathbf{r}_{l_m} \mathbf{k}^{(m)} \right\}, \\ A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) &= e^{i\mathbf{r}_{l_1} \mathbf{k}'} e^{i\mathbf{r}_{l_m} \mathbf{k}^{(m)}} A_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{k}, \vec{k}). \end{aligned} \quad (7.12)$$

According to (3.10), $\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}) = \mathbf{k} - \mathbf{k}' - \dots - \mathbf{k}^{(m-1)}$. Hence, picking a vector \mathbf{p} with a unit length, we obtain the formula

$$\exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) = \frac{\varrho \mathbf{p} \cdot \nabla_{\mathbf{k}'} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})}{i[\mathbf{p} \cdot \nabla_{\mathbf{k}'} \varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \tau_1 - \varrho \mathbf{p} \cdot (\mathbf{r}_{l_1} - \mathbf{r}_{l_m})]}. \quad (7.13)$$

If we set

$$\begin{aligned} \varphi' &= \nabla_{\mathbf{k}'} \varphi_{\zeta, \vec{\zeta}}(\mathbf{k}_{*l}, \vec{k}_{*}) = \nabla_{\mathbf{k}'} \omega(\zeta' \mathbf{k}'_{*}) - \nabla_{\mathbf{k}^{(m)}} \omega(\zeta^{(m)} \mathbf{k}_{*}^{(m)}), \\ c_p &= \mathbf{p} \cdot \varphi', q_p = \varrho \mathbf{p} \cdot (\mathbf{r}_{l_1} - \mathbf{r}_{l_m}), \end{aligned} \quad (7.14)$$

$$\theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1) = \frac{(c_p \tau_1 - q_p)}{[\mathbf{p} \cdot \nabla_{\mathbf{k}'} \varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) \tau_1 - \mathbf{p} \cdot (\mathbf{r}_{l_1} - \mathbf{r}_{l_m})]}, \quad (7.15)$$

then (7.13) can be recast as

$$\exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) = \frac{\varrho \mathbf{p} \cdot \nabla_{\mathbf{k}'} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})}{i(c_p \tau_1 - q_p)} \theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1). \quad (7.16)$$

If (2.51) holds, then $\varphi' \neq 0$, and to get $|c_p| \neq 0$, we can take

$$\mathbf{p} = |\varphi'|^{-1} \cdot \varphi', \quad |c_p| = p_0 > 0. \quad (7.17)$$

If (2.54) holds, we have $\varphi' = 0$, and we set

$$\mathbf{p} = |(\mathbf{r}_{l_1} - \mathbf{r}_{l_m})|^{-1} \cdot (\mathbf{r}_{l_1} - \mathbf{r}_{l_m}). \quad (7.18)$$

Let consider first the case, where (2.51) holds. Notice that the denominator in (7.16) vanishes for

$$\tau_{10} = \frac{q_p}{c_p}. \quad (7.19)$$

We split the integral with respect to τ_1 in (7.11) into the sum of two integrals, namely

$$\mathcal{F}_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{w}_{l_1} \dots \mathbf{w}_{l_m})(\mathbf{k}, \tau) = F_1 + F_2, \quad (7.20)$$

$$F_1 = \int_{|\tau_{10} - \tau_1| \geq c_0 \beta^{1-\varepsilon} |\ln \beta|} \int_{\mathbb{D}_m} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) \times A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) \tilde{d}^{(m-1)} \vec{k} d\tau_1,$$

$$F_2 = \int_{|\tau_{10} - \tau_1| < c_0 \beta^{1-\varepsilon} |\ln \beta|} \int_{\mathbb{D}_m} \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) \times A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m}) \tilde{d}^{(m-1)} \vec{k} d\tau_1,$$

where c_0 is a large enough constant which we estimate below in (7.28). Since \mathbf{w}_j are bounded in E and (2.48) holds, we obtain similarly to (4.10) the estimate

$$\|F_2\|_{L^1} \leq C c_0 \beta^{1-\varepsilon} |\ln \beta| \prod_{j=1}^m \|\mathbf{w}_{l_j}\|_E \leq C_1(R) \frac{\varrho |\ln \beta|}{\beta^{1+\varepsilon}}. \quad (7.21)$$

To estimate the norm of F_1 , we use (7.13) and integrate by parts the integral in (7.20) with respect to \mathbf{k}' . We obtain

$$F_1 = \int_{|\tau_{10} - \tau_1| \geq \beta^{1-\varepsilon} |\ln \beta|} I(\mathbf{k}, \tau_1) d\tau_1, \tag{7.22}$$

$$I(\mathbf{k}, \tau_1) = - \int_{\mathbb{D}_m} \frac{\varrho \exp_{\varphi}(\mathbf{k}, \vec{k}, \tau_1, \varrho, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})}{i(c_p \tau_1 - \varrho q_p)} \times \mathbf{p} \cdot \nabla_{\mathbf{k}'} [\theta_0 A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})] \tilde{\mathbf{d}}^{(m-1)d} \vec{k}.$$

According to (7.10) and (3.10), the expansion of the gradient $\nabla_{\mathbf{k}'}$ in the above formula involves the derivatives of $\chi, \theta_0, e^{i\mathbf{r}_{l_1} \mathbf{k}' \cdot \mathbf{w}_{l_1}}$ and $e^{i\mathbf{r}_{l_m} \mathbf{k}' \cdot \mathbf{w}_{l_m}}$. To estimate θ_0 and $\nabla \theta_0$, we note that

$$\theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1) = \frac{(\mathbf{p} \cdot \varphi' \tau_1 - q_p)}{(\mathbf{p} \cdot \varphi' \tau_1 - q_p) + \tau_1 \mathbf{p} \cdot [\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi']} = \frac{1}{1 + \tau_1 \mathbf{p} \cdot [\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi'] / (c_p \tau_1 - q_p)}. \tag{7.23}$$

Since $|\tau_{10} - \tau_1| \geq c_0 \beta^{1-\varepsilon} |\ln \beta|$, from (7.19) we infer

$$|c_p \tau_1 - q_p| \geq c_p c_0 \beta^{1-\varepsilon} |\ln \beta|. \tag{7.24}$$

From (6.32) we see that in the integral (7.22) the integrands are nonzero only if

$$|\mathbf{k}^{(j)} - \zeta^{(j)} \mathbf{k}_*^{(j)}| \leq \pi_0 \beta^{1-\varepsilon}, |\mathbf{k} - \zeta \mathbf{k}_*| \leq m \pi_0 \beta^{1-\varepsilon}, \tag{7.25}$$

where $\pi_0 \leq 1$. Using the Taylor remainder estimate for $\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}$ at \vec{k}_* , we obtain the inequality

$$|\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi'| \leq 2m C_{\omega, 2} \beta^{1-\varepsilon}. \tag{7.26}$$

Hence in (7.23)

$$|\tau_1 \mathbf{p} \cdot [\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi'] / (c_p \tau_1 - q_p)| \leq 2m \tau_* C_{\omega, 2} / (c_p c_0 |\ln \beta|). \tag{7.27}$$

Suppose that $\beta \leq 1/2$ is small and c_0 satisfies

$$\frac{m \tau_* C_{\omega, 2}}{|\ln \beta|} \leq \frac{m \tau_* C_{\omega, 2}}{\ln 2} \leq \frac{1}{4} |c_p| c_0. \tag{7.28}$$

Then it follows from (7.23) with the help of (7.28), (3.2), (7.24) and (7.27) that

$$|\theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1)| \leq 2. \tag{7.29}$$

Obviously,

$$\nabla_{\mathbf{k}'}\theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1) = \frac{-\tau_1 \nabla_{\mathbf{k}'} [\mathbf{P} \cdot (\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi')]}{(c_p \tau_1 - q_p) [1 + \tau_1 \mathbf{P} \cdot (\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi') / (c_p \tau_1 - q_p)]^2}. \quad (7.30)$$

Using (7.27), (7.28), and (3.2), we obtain

$$|\nabla_{\mathbf{k}'}\theta_0(\mathbf{k}, \vec{k}, \varrho, \tau_1)| \leq \frac{4\tau_*}{|c_p \tau_1 - q_p|} |\nabla_{\mathbf{k}'} [\mathbf{P} \cdot (\nabla_{\mathbf{k}'} \varphi_{\zeta, \bar{\zeta}}(\mathbf{k}, \vec{k}) - \varphi')]| \leq \frac{4\tau_* C_{\omega, 2}}{|c_p \tau_1 - q_p|}. \quad (7.31)$$

To estimate $\nabla_{\mathbf{k}'}\chi$, we use (3.13). We conclude that the absolute value of the integral (7.22) is not greater than

$$\begin{aligned} |I(\mathbf{k}, \tau_1)| &\leq \frac{4\varrho\tau_* C_{\omega, 2}}{|\tau_1 c_p - q_p|^2} \int_{\mathbb{D}_m} |A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})| \tilde{d}^{(m-1)d} \vec{k} \\ &\quad + \frac{2\varrho\tau_*}{|\tau_1 c_p - q_p|} \int_{\mathbb{D}_m} [|\nabla_{\mathbf{k}'} A(\mathbf{k}, \vec{k}, \mathbf{r}_{l_1}, \mathbf{r}_{l_m})|] \tilde{d}^{(m-1)d} \vec{k} \\ &\leq \left[\frac{4C_{\omega, 2}\varrho\tau_*}{|\tau_1 c_p - q_p|^2} \|\chi^{(m)}(\mathbf{k}, \cdot)\| + \frac{2\varrho\tau_*}{|\tau_1 c_p - q_p|} \|(\nabla_{\mathbf{k}'} - \nabla_{\mathbf{k}^{(m)}})\chi^{(m)}(\mathbf{k}, \cdot)\| \right] \\ &\quad \times \prod_{j=1}^m \|\mathbf{w}_j\|_{L^1} + \frac{2\varrho\tau_* \|\chi^{(m)}(\mathbf{k}, \cdot)\|}{|\tau_1 c_p - q_p|} \left[\prod_{j=2}^m \|\mathbf{w}_{l_j}\|_{L^1} \|\nabla_{\mathbf{k}'} e^{i\mathbf{r}_{l_1} \mathbf{k}'} \mathbf{w}_{l_1}\|_{L^1} \right. \\ &\quad \left. + \prod_{j=1}^{m-1} \|\mathbf{w}_j\|_{L^1} \|\nabla_{\mathbf{k}^{(m)}} e^{i\mathbf{r}_{l_m} \mathbf{k}^{(m)}} \mathbf{w}_m\|_{L^1} \right]. \quad (7.32) \end{aligned}$$

Note that $\|\mathbf{w}_j\|_{L^1}$ are bounded according to (2.27) and $\nabla_{\mathbf{k}^{(m)}} e^{i\mathbf{r}_{l_m} \mathbf{k}^{(m)}} \mathbf{w}_{l_m}$, $\nabla_{\mathbf{k}'} e^{i\mathbf{r}_{l_1} \mathbf{k}'} \mathbf{w}_{l_1}$ by (2.33). Hence we obtain

$$|I(\mathbf{k}, \tau_1)| \leq \frac{C_2 \varrho \beta^{-1-\varepsilon}}{\tau_1 c_p - q_p} + \frac{\varrho C_2}{|\tau_1 c_p - q_p|^2}. \quad (7.33)$$

Obviously,

$$\begin{aligned} \int_{|\tau_1 - q_p/c_p| \geq c_0 \beta^{1-\varepsilon} |\ln \beta|} |\tau_1 c_p - q_p|^{-1} d\tau_1 &= \frac{1}{c_p} \int_{c_0 \beta^{1-\varepsilon} |\ln \beta|}^{\tau_* - q_p/c_p} \frac{d\tau_1}{\tau_1} \\ &= \frac{1}{c_p} \ln \frac{\tau_* - q_p/c_p}{c_0 \beta^{1-\varepsilon} |\ln \beta|} \leq \frac{1}{c_p} (C + |\ln[\beta^{1-\varepsilon} |\ln \beta|]|) \\ &\leq \frac{1}{c_p} [C + |\ln \beta| + |\ln |\ln \beta||] \leq \frac{1}{c_p} [C + 2 |\ln \beta|]. \end{aligned}$$

Similarly, using (2.48), we get

$$\begin{aligned} & \int_{|\tau_1 - q_p/c_p| \geq c_0 \beta^{1-\varepsilon} |\ln \beta|} |\tau_1 c_p - q_p|^{-2} d\tau_1 \\ &= \frac{1}{c_p} \int_{c_0 \beta^{1-\varepsilon} |\ln \beta|}^{\tau_* - q_p/c_p} \frac{d\tau_1}{\tau_1^2} = \frac{1}{c_p} \left[\frac{1}{c_0 \beta^{1-\varepsilon} |\ln \beta|} - \frac{1}{\tau_* - q_p/c_p} \right] \\ &\leq \frac{1}{c_p c_0 \beta^{1-\varepsilon} |\ln \beta|} \leq \frac{C_3 \varrho}{\beta^{-1-\varepsilon} |\ln \beta|}. \end{aligned}$$

Hence we obtain for small β

$$\|\mathcal{F}_{\zeta, \vec{\zeta}}^{(m)}(\mathbf{w}_1 \dots \mathbf{w}_m)(\mathbf{k}, \tau)\|_E \leq C_4 \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|. \tag{7.34}$$

Now let us consider the case, where (2.54) holds, $\varphi' = 0$ and \mathbf{p} is defined by (7.18). Turning to expression (7.23), we notice that

$$c_p \tau_1 - q_p = -\varrho |\mathbf{r}_{l_1} - \mathbf{r}_{l_m}|, \quad \tau_* |c_p \tau_1 - q_p|^{-1} \leq \frac{1}{\beta^{1+\varepsilon}},$$

and, according to (7.26),

$$|\nabla_{\mathbf{k}'} \varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) - \varphi'| \leq C_{\omega, 2} \beta^{1-\varepsilon}.$$

Then we estimate the denominator in (7.23) and (7.30) using (2.54):

$$|\tau_1 \mathbf{p} \cdot [\nabla_{\mathbf{k}'} \varphi_{\zeta, \vec{\zeta}}(\mathbf{k}, \vec{k}) - \varphi'] / (c_p \tau_1 - q_p)| \leq \tau_* C_{\omega, 2} \beta^{1-\varepsilon} / (\varrho |\mathbf{r}_{l_1} - \mathbf{r}_{l_m}|) \leq \frac{1}{2}.$$

If β is so small that (7.28) holds, we again get (7.29) and (7.31). Hence we obtain (7.34) in this case as well (in fact, in this case, the logarithmic factor can be omitted). Finally, we obtain (7.35) from (7.34) after summing up over all $\vec{\lambda}, \vec{\zeta}$. □

Lemma 7.2. *Let the nk -spectrum S be universally resonance invariant. Let the operators $\mathcal{F}_{\text{av}, n_l, \vartheta}(\tilde{\mathbf{w}})$, $\mathcal{F}_{\text{av}, n_l, \vartheta, \text{diag}}(\tilde{\mathbf{v}})$, and $\mathcal{F}_{\text{av}, n_l, \vartheta, \text{coup}}$ be defined respectively by (6.5), (4.7), and (7.4). Let $\tilde{\mathbf{v}}, \|\tilde{\mathbf{v}}\|_{E^N} \leq 2R$, be a multi-wavepacket solution of (6.9) with the nk -spectrum S . Then for small β and ϱ*

$$\|\mathcal{F}_{\text{av}, n_l, \vartheta, \text{coup}}(\tilde{\mathbf{v}})\|_{E^N} \leq C \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|. \tag{7.35}$$

PROOF. According to (6.5), (7.1), and (7.4), $\mathcal{F}_{\text{av}, n_l, \vartheta, \text{coup}}$ involves only terms with $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m \setminus \Lambda_{n_l, \vartheta}^{m, \text{diag}}$ and it is sufficient to prove the estimate (7.8) for indices $\vec{\lambda} \in \Lambda_{n_l, \vartheta}^m \setminus \Lambda_{n_l, \vartheta}^{m, \text{diag}}$. Such indices involve at least two

components $\lambda_i = (\zeta_i, l_i)$ and $\lambda_j = (\zeta_j, l_j)$ with $l_i \neq l_j$ since the nk -spectrum is universally invariant, see (3.26). According to Theorem 6.13, the solution $\tilde{\mathbf{v}}$ is a particle-like wavepacket. Therefore, all the components of $\tilde{\mathbf{v}}_\chi$ are particle-like; (6.32) holds according to (5.11). Hence all the conditions of Lemma 7.1 are fulfilled and (7.35) follows from (7.8). \square

Note now that every equation (7.7) is an approximation of Equation (4.6) with single-wavepacket initial data $\hat{\mathbf{h}}_l$, namely

$$\hat{\mathbf{u}}_l(\mathbf{k}, \tau) = \mathcal{F}(\hat{\mathbf{u}}_l)(\mathbf{k}, \tau) + \hat{\mathbf{h}}_l(\mathbf{k}). \tag{7.36}$$

One can apply to this equation Theorems 5.6 and 6.4 formally restricted to the case $N = 1$ of a single wavepacket. Based on this observation and above lemma, we prove the following theorem which implies previously formulated Theorems 2.14 and 2.15.

Theorem 7.3. *Assume that the multi-wavepacket $\tilde{\mathbf{h}} = \sum \hat{\mathbf{h}}_l$ is particle-like and its nk -spectrum is universally resonance invariant. Assume also that either (2.51) or (2.54) holds. Let $\hat{\mathbf{u}}$ be a solution of Equation (4.6). Let $\hat{\mathbf{u}}_l$ be solutions of (7.36). Then the superposition principle holds, namely*

$$\left\| \hat{\mathbf{u}} - \sum_{l=1}^N \hat{\mathbf{u}}_l \right\| \leq C \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C\beta^s. \tag{7.37}$$

PROOF. Let $\mathbf{v}_{\text{diag},l}$ be a solution of the decoupled system (7.7). We compare the systems (7.5) and (7.6). The difference between the systems is the term $\mathcal{F}_{\text{av},n_l,\vartheta,\text{coup}}(\tilde{\mathbf{v}})$. According to Theorem 6.12, the solution $\tilde{\mathbf{v}}$ is a particle-like wavepacket and we can apply Lemma 7.2. According to this lemma, (7.35) holds. Applying Lemma 4.6 to Equations (7.5) and (7.6) and using (7.35), we conclude that the difference of their solutions satisfies the inequality

$$\|\mathbf{v}_l - \mathbf{v}_{\text{diag},l}\|_E \leq C' \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C'\beta^s. \tag{7.38}$$

According to Theorem 6.13, the inequality (6.55) holds, where $\tilde{\mathbf{v}}$ is a solution of (6.9) which can be rewritten in the form of (7.5). From (6.55) and (7.38) we infer

$$\left\| \hat{\mathbf{u}} - \sum_{l=1}^N \mathbf{v}_{\text{diag},l} \right\| \leq C_1 \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C_1\beta^s. \tag{7.39}$$

Note that Equation (7.7) for $\mathbf{v}_{\text{diag},l}$ coincides with the averaged equation (6.9) obtained for the wave interaction system derived for (7.36). Therefore, applying Theorems 5.6 and 6.4 to the case $N = 1$ and $\hat{\mathbf{h}} = \hat{\mathbf{h}}_l$, we

deduce from (5.49) and (6.11) the estimate

$$\|\hat{\mathbf{u}}_l - \mathbf{v}_{\text{diag},l}\|_E \leq C_2 \varrho + C'_2 \beta^s. \tag{7.40}$$

Finally, from (7.39) and (7.40) we infer (7.37). □

7.1. Generalizations.

In this section, we show that the particle-like wavepacket invariance can be extended to the case, where nk -spectra S are not universally resonance invariant. So, suppose that an nk -spectrum S is resonance invariant and consider nonlinearities of the form similar to (6.5)

$$\mathcal{F}_{\text{res},n_l,\vartheta}(\tilde{\mathbf{w}}) = \sum_{m \in \mathfrak{M}_F} \mathcal{F}_{n_l,\vartheta}^{(m)}, \quad \mathcal{F}_{n_l,\vartheta}^{(m)} = \sum_{\vec{\lambda} \in \Lambda'_{n_l,\vartheta}} \mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \tag{7.41}$$

where $\Lambda'_{n_l,\vartheta} \subseteq \Lambda_{n_l,\vartheta}^m$ is a given subset of Λ^m . Obviously, \mathcal{F}_{av} defined by (6.5) has the form of (7.41) with $\Lambda'_{n_l,\vartheta} = \Lambda_{n_l,\vartheta}^m$. Let us introduce a multi-wavepacket

$$\tilde{\mathbf{w}} = (\mathbf{w}_{n_1,+}, \mathbf{w}_{n_1,-}, \dots, \mathbf{w}_{n_N,+}, \mathbf{w}_{n_N,-}) \tag{7.42}$$

with the nk -spectrum $S = \{(n_l, \theta), l = 1, \dots, N; \theta = \pm\}$.

We call a subset $S' \subset S$ *sign-invariant* if when it has (n_l, θ) as an element, then $(n_l, -\theta)$ is also its element. Suppose that $S' \subset S$ is sign-invariant. It is easy to see that if a set $S' \subset S$ is sign-invariant, then it is uniquely defined by a subset of indices $I' = I'(S') \subset I = \{1, \dots, N\}$, namely

$$S' = \{(n_l, \theta) : l \in I'(S'), \theta = \pm\}.$$

Definition 7.4. We call an index pair (n_l, \mathbf{k}_{*l}) *Group Velocity Matched (GVM)* with $\mathcal{F}_{\text{res},n_l,\vartheta}$ if every nonzero term $\mathcal{F}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}^{(m)}$ in the sum (7.41) has an index $\vec{\lambda}$ such that for at least one component $\lambda_j = (\zeta^{(j)}, l_j)$ of this index the following equality holds:

$$\nabla \omega_{n_l}(\mathbf{k}_{*l}) = \nabla \omega_{n_{l_j}}(\mathbf{k}_{*l_j}). \tag{7.43}$$

We call S' a *GVM set* with respect to the nonlinearity \mathcal{F}_{res} defined by (7.41) if $S' \subset S$ is sign-invariant and every $(n_l, \mathbf{k}_{*l}) \in S'$ is GVM.

Obviously, if S is universally resonance invariant and $\Lambda'_{n_l,\vartheta} = \Lambda_{n_l,\vartheta}^m$ as in (6.5), then S is a GVM set and, in this case, $l_j = I_0$ as in Definition 3.6. If $S' \subset S$ is sign-invariant, we call a multi-wavepacket $\tilde{\mathbf{w}}$ as in (7.42) with the

nk -spectrum $S = \{(n_l, \theta), l = 1, \dots, N; \theta = \pm\}$ partially S' -localized multi-wavepacket if for every $(n_l, \theta) \in S'$ the wavepacket $\mathbf{w}_{n_l, \theta}$ is a spatially localized with position \mathbf{r}_{*l} . Note that, according to Definition 2.7, if $S' = S$ and \mathbf{w} is a partially S' -localized multi-wavepacket, then it is a multi-particle wavepacket.

Theorem 2.10 on the particle-like wavepacket preservation can be generalized as follows.

Theorem 7.5 (preservation of spatially localized wavepackets). *Assume that the conditions of Theorem 2.9 hold, in particular the initial datum $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta, \mathbf{k})$ is a multi-wavepacket with an nk -spectrum S . Assume also that $S' \subset S$ is a GVM set, $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta, \mathbf{k})$ is partially S' -localized wavepacket with positions \mathbf{r}_{*l} , $l \in I'(S')$, and (2.47) holds. Then the solution $\hat{\mathbf{u}}(\tau, \beta) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ to (2.14) for any $\tau \in [0, \tau_*]$ is a multi-wavepacket with the nk -spectrum S and it is an S' -localized wavepacket with positions \mathbf{r}_{*l} , $l \in I'(S')$. Namely, (2.46) holds, where $\hat{\mathbf{u}}_l$ is a wavepacket with the nk -pair $(n_l, \mathbf{k}_{*l}) \in S'$ defined by (2.45), the constants C, C_1, C_2 do not depend on \mathbf{r}_{*l} , and every $\hat{\mathbf{u}}_l$, $l \in I'(S')$, is equivalent in the sense of the equivalence (2.42) of degree $s_1 = \min(s, s_0)$ to a spatially localized wavepacket with position \mathbf{r}_{*l} .*

PROOF. The proof of the theorem is the same as the proof of Theorem 2.10 since it used only the fact that a universally resonance invariant set is a GVM one, that allows us to apply Lemma 6.9. One also have to use the space $(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')$ with the norm defined by the formula similar to (6.46):

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')} = \sum_{l, \vartheta} \|\hat{\mathbf{w}}_{l, \vartheta}\|_E + \beta^{1+\varepsilon} \sum_{\vartheta=\pm} \sum_{l \in I'(S')} \|\nabla_{\mathbf{k}}(e^{-i\mathbf{r}_{*l}\mathbf{k}} \hat{\mathbf{w}}_{l, \vartheta})\|_E. \tag{7.44}$$

After replacing $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ with $(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')$ we can literally repeat all the steps of the proof of Theorem 2.10 and obtain the statement of Theorem 7.5. \square

Below we prove that the superposition principle can hold not only for universal resonance invariant multi-wavepackets, but for other cases allowing resonant processes such as the second and third harmonic generations, three-wave interaction, etc. Here we prove a theorem applicable to such situations, which is more general than Theorem 2.14.

Let us consider a multi-wavepacket with a resonance invariant nk -spectrum

$$S = \{(n_l, \mathbf{k}_{*l}), l = 1, \dots, N\}$$

as in (3.14), and assume that is the union of spectra S_p :

$$S = S_1 \cup \dots \cup S_K, S_p \cap S_q = \emptyset \text{ if } p \neq q. \quad (7.45)$$

Recall that resonance interactions are defined in terms of vectors $\vec{\lambda} \in \Lambda^m$ (see (3.16), (3.17)). We call a vector $\vec{\lambda} = ((\zeta', l_1), \dots, (\zeta^{(m)}, l_m)) \in \Lambda^m$ a *cross-interacting (CI)* if there exist at least two indices $(\zeta^{(i)}, l_i)$ and $(\zeta^{(j)}, l_j)$ such that $(\zeta^{(i)}, l_i) \in S_{p_i}, (\zeta^{(j)}, l_j) \in S_{p_j}$ with $p_i \neq p_j$.

Definition 7.6 (partially GVM decomposition). We call the decomposition (7.45) *partially GVM* with respect to \mathcal{F}_{res} defined by (7.41) if the following two conditions are satisfied: (i) every spectrum $S_j, j = 1, \dots, K$, is resonance invariant; (ii) a solution $(m, \zeta, n, \vec{\lambda}) \in P(S)$ of the resonance equation (3.24) with CI vector $\vec{\lambda} = ((\zeta', l_1), \dots, (\zeta^{(m)}, l_m))$ has at least two indices $(\zeta^{(i)}, l_i) \in S_{p_i}$ and $(\zeta^{(j)}, l_j) \in S_{p_j}$ with $p_i \neq p_j$ such that both l_i and l_j are GVM with respect to \mathcal{F}_{res} and

$$|\nabla_{\mathbf{k}} \omega_{n_l}(\mathbf{k}_{*l_i}) - \nabla_{\mathbf{k}} \omega_{n_l}(\mathbf{k}_{*l_j})| \neq 0. \quad (7.46)$$

Now we use Lemma 7.1 for small coupling. Being given a partially GVM decomposition (7.45), we introduce the set of coupling terms between S_{p_i} and S_{p_j} as follows:

$$\Lambda_{n_l, \vartheta}^{m, \text{coup}} = \left\{ \vec{\lambda} = (\vec{l}, \vec{\zeta}) \in \Lambda_{n_l, \vartheta}^m : \exists i \neq j \text{ such that } l_i \in S_{p_i}, l_j \in S_{p_j} \right\}, \quad (7.47)$$

We also introduce a set of interactions reducible to every S_p (block-diagonal) which is similar to (7.2):

$$\Lambda_{n_l, \vartheta}^{m, \text{red}} = \Lambda_{n_l, \vartheta}^m \setminus \Lambda_{n_l, \vartheta}^{m, \text{coup}}, \quad (7.48)$$

and the reduced operator

$$\begin{aligned} \mathcal{F}_{\text{av}, n_l, \vartheta, \text{red}}(\tilde{\mathbf{w}}) &= \sum_{m \in \mathcal{M}_F} \mathcal{F}_{n_l, \vartheta, \text{red}}^{(m)}(\tilde{\mathbf{w}}), \mathcal{F}_{n_l, \vartheta, \text{red}}^{(m)}(\tilde{\mathbf{w}}) \\ &= \sum_{\vec{\lambda} \in \Lambda_{n_l, \vartheta}^{m, \text{red}}} \mathcal{F}_{n_l, \vartheta, \vec{\xi}(\vec{\lambda})}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \end{aligned} \quad (7.49)$$

where $\Lambda_{n_l, \vartheta}^{m, \text{red}}$ is defined by (7.48). Note that if the set S is universal resonance invariant and every S_{p_i} is a two-point set $\{(+, l_i), (+, l_i)\}$, then

$\Lambda_{n_l, \vartheta}^{m, \text{red}} = \Lambda_{n_l, \vartheta}^{m, \text{diag}}$. We introduce also a partially decoupled, reduced system similar to (7.6)

$$\tilde{\mathbf{v}}_{\text{red}} = \mathcal{F}_{\text{av}, \Psi, \text{red}}(\tilde{\mathbf{v}}_{\text{red}}) + \tilde{\mathbf{h}}_{\Psi}, \quad (7.50)$$

which can be rewritten in the decoupled form similar to (7.7):

$$\mathbf{v}_{\text{red}, p} = \mathcal{F}_{\text{av}, \Psi, \text{red}, p}^{(m)}(\mathbf{v}_{\text{red}, p}) + \mathbf{h}_{\text{red}, \Psi, p}, \quad p = 1, \dots, K. \quad (7.51)$$

Now $\mathbf{v}_{\text{red}, p}$ may include more than one wavepacket, namely

$$\mathbf{v}_{\text{red}, p} = \sum_{(n_l, \theta) \in S_p} (\tilde{\mathbf{v}}_{\text{red}})_{n_l, \theta}, \quad \mathbf{h}_{\text{red}, \Psi, p} = \sum_{(n_l, \theta) \in S_p} (\tilde{\mathbf{h}}_{\Psi})_{n_l, \theta}, \quad p = 1, \dots, K. \quad (7.52)$$

The following theorem is a generalization of Theorem 2.14 on the superposition.

Theorem 7.7 (general superposition principle). *Suppose that the initial data $\hat{\mathbf{h}}$ of (2.14) is a multi-wavepacket of the form*

$$\hat{\mathbf{h}} = \sum_{p=1}^K \hat{\mathbf{h}}_{\text{red}, p}, \quad (7.53)$$

where $\hat{\mathbf{h}}$ is a multi-wavepacket in the sense of Definition 3.8 with a resonance invariant nk -spectrum S , $\hat{\mathbf{h}}_{\text{red}, p}$ is a multi-wavepacket with a resonance invariant nk -spectrum S_p , and the decomposition (7.45) is a partially GVM in the sense of Definition 7.6 with respect to the nonlinearity \mathcal{F}_{av} defined by (6.5). Suppose also that (2.48) holds. Then the solution $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ to the evolution equation (2.14) satisfies the approximate superposition principle

$$\mathcal{G}\left(\sum_{p=1}^K \hat{\mathbf{h}}_{\text{red}, p}\right) = \sum_{p=1}^K \mathcal{G}(\hat{\mathbf{h}}_{\text{red}, p}) + \tilde{\mathbf{D}}, \quad (7.54)$$

with a small remainder $\tilde{\mathbf{D}}(\tau)$ satisfying the following estimate:

$$\sup_{0 \leq \tau \leq \tau_*} \|\tilde{\mathbf{D}}(\tau)\|_{L^1} \leq C_{\varepsilon} \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|, \quad (7.55)$$

where ε is the same as in Definition 2.1 and can be arbitrary small, τ_* does not depend on β , ϱ , and ε .

PROOF. The proof of Theorem 7.7 is similar to the proof of Theorem 7.3. The averaged system (6.9) can be written similarly to (7.5) in the form

$$\tilde{\mathbf{v}} = \mathcal{F}_{\text{av}, \Psi, \text{red}}(\tilde{\mathbf{v}}) + \mathcal{F}_{\text{av}, \Psi, \text{coup}}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \quad (7.56)$$

Comparing now the systems (7.56) and (7.50), we find that the difference between them is the term $\mathcal{F}_{\text{av}, n_l, \vartheta, \text{coup}}(\tilde{\mathbf{v}})$. According to Theorem 7.5, the

solution $\tilde{\mathbf{v}}$ is a spatially localized wavepacket and hence we can apply Lemma 7.2 getting the inequality (7.35). Applying Lemma 4.6 to Equations (7.56) and (7.50) and using (7.35), we conclude that the difference of their solutions satisfies the inequality

$$\|\mathbf{v}_p - \mathbf{v}_{\text{red},p}\|_E \leq C' \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C' \beta^s, p = 1, \dots, K \quad (7.57)$$

According to Theorem 6.13, the inequality (6.55) holds, where $\tilde{\mathbf{v}}$ is a solution of (7.56), and we infer from (7.57)

$$\left\| \hat{\mathbf{u}} - \sum_{p=1}^K \mathbf{v}_{\text{red},p} \right\| \leq C_1 \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C_1 \beta^s. \quad (7.58)$$

Similarly to (7.36) we introduce equation for $\hat{\mathbf{u}}_{\text{red},p} = \mathcal{G}(\hat{\mathbf{h}}_{\text{red},p})$

$$\hat{\mathbf{u}}_{\text{red},p}(\mathbf{k}, \tau) = \mathcal{F}(\hat{\mathbf{u}}_{\text{red},p})(\mathbf{k}, \tau) + \hat{\mathbf{h}}_{\text{red},p}(\mathbf{k}). \quad (7.59)$$

Applying Theorems 5.6 and 6.4, we infer similarly to (7.40) the inequality

$$\|\hat{\mathbf{u}}_{\text{red},p} - \mathbf{v}_{\text{red},p}\|_E \leq C_2 \varrho + C_2' \beta^s. \quad (7.60)$$

Finally, from (7.58) and (7.60) we infer (7.55). \square

Acknowledgment: Effort of A. Babin and A. Figotin is sponsored by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant number FA9550-04-1-0359.

References

1. A. Babin and A. Figotin, *Nonlinear Photonic Crystals: I. Quadratic nonlinearity*, *Waves Random Media* **11** (2001), no. 2, R31-R102.
2. A. Babin and A. Figotin, *Nonlinear Photonic Crystals: II. Interaction classification for quadratic nonlinearities*, *Waves Random Media* **12** (2002), no. 4, R25-R52.
3. A. Babin and A. Figotin, *Nonlinear Photonic Crystals: III. Cubic Nonlinearity*, *Waves Random Media* **13** (2003), no. 4, R41-R69.
4. A. Babin and A. Figotin, *Nonlinear Maxwell Equations in Inhomogeneous Media*, *Commun. Math. Phys.* **241** (2003), 519-581.
5. A. Babin and A. Figotin, *Polylinear spectral decomposition for nonlinear Maxwell equations*, In: *Partial Differential Equations* (M. S. Agranovich and M.A. Shubin, Eds.), *Am. Math. Soc. Translations. Series 2* **206**, 2002, pp. 1-28.

6. A. Babin and A. Figotin, *Nonlinear Photonic Crystals: IV Nonlinear Schrödinger Equation Regime*, Waves Random Complex Media **15** (2005), no. 2, 145-228.
7. A. Babin and A. Figotin, *Wavepacket Preservation under Nonlinear Evolution*, arXiv:math.AP/0607723.
8. A. Babin and A. Figotin, *Linear superposition in nonlinear wave dynamics*, Reviews Math. Phys. **18** (2006), no. 9, 971-1053.
9. D. Bambusi, *Birkhoff normal form for some nonlinear PDEs*, Commun. Math. Phys. **234** (2003), no. 2, 253-285.
10. W. Ben Youssef and D. Lannes, *The long wave limit for a general class of 2D quasilinear hyperbolic problems*, Commun. Partial Differ. Equations **27** (2002), no. 5-6, 979-1020.
11. N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Hindustan Publ. Corp. Delhi, 1961.
12. J. L. Bona, T. Colin, and D. Lannes, *Long wave approximations for water waves*, Arch. Ration. Mech. Anal. **178** (2005), no. 3, 373-410.
13. J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations*, Am. Math. Soc., Providence, RI, 1999.
14. T. Cazenave, *Semilinear Schrödinger Equations*, Am. Math. Soc., Providence, RI, 2003.
15. T. Colin, *Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems*, Asymp. Anal. **31** (2002), no. 1, 69-91.
16. T. Colin and D. Lannes, *Justification of and long-wave correction to Davey-Stewartson systems from quadratic hyperbolic systems.*, Discr. Cont. Dyn. Syst. **11** (2004), no. 1, 83-100.
17. W. Craig and M. D. Groves, *Normal forms for wave motion in fluid interfaces*, Wave Motion **31** (2000), no. 1, 21-41.
18. W. Craig, C. Sulem, and P.-L. Sulem, *Nonlinear modulation of gravity waves: a rigorous approach*, Nonlinearity **5** (1992), no. 2, 497-522.
19. T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on \mathbf{R}^2* , Arch. Ration. Mech. Anal. **163** (2002), no. 3, 209-258.
20. Zh. Gang and I. M. Zhou, *On soliton dynamics in nonlinear Schrödinger equations*, Geom. Funct. Anal. **16** (2006), no. 6, 1377-1390.
21. Zh. Gang and I. M. Zhou, *Relaxation of Solitons in Nonlinear Schrodinger Equations with potential*, arXiv:math-ph/0603060v1

22. J. Giannoulis and A. Mielke, *The nonlinear Schrödinger equation as a macroscopic limit for an oscillator chain with cubic nonlinearities*, Nonlinearity **17** (2004), no. 2, 551–565.
23. G. Iooss and E. Lombardi, *Polynomial normal forms with exponentially small remainder for analytic vector fields*. J. Differ. Equations **212** (2005), no. 1, 1–61.
24. J.-L. Joly, G. Metivier, and J. Rauch, *Diffractive nonlinear geometric optics with rectification*, Indiana Univ. Math. J. **47** (1998), no. 4, 1167–1241.
25. B. L. G. Jonsson, J. Fröhlich, S. Gustafson, and I. M. Sigal, *Long time motion of NLS solitary waves in a confining potential*, Ann. Henri Poincaré **7** (2006), no. 4, 621–660.
26. L. A. Kalyakin, *Long-wave asymptotics. Integrable equations as the asymptotic limit of nonlinear systems*, Russian Math. Surv. **44** (1989), no. 1, 3–42.
27. L. A. Kalyakin, *Asymptotic decay of a one-dimensional wave packet in a nonlinear dispersion medium*, Math. USSR Sb. Surveys **60** (2) (1988) 457–483.
28. T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1980.
29. S. B. Kuksin, *Fifteen years of KAM for PDE*, In: Geometry, Topology, and Mathematical Physics, Am. Math. Soc., Providence, RI, 2004, pp. 237–258.
30. P. Kirrmann, G. Schneider, and A. Mielke, *The validity of modulation equations for extended systems with cubic nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **122** (1992), no. 1-2, 85–91.
31. J. Krieger and W. Schlag, *Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension*, J. Am. Math. Soc. **19** (2006), no. 4, 815–920 (electronic).
32. V. P. Maslov, *Non-standard characteristics in asymptotic problems*, Russian Math. Surv. **38** (1983), 6, 1–42.
33. A. Mielke, G. Schneider, and A. Ziegra, *Comparison of inertial manifolds and application to modulated systems*, Math. Nachr. **214** (2000), 53–69.
34. R. D. Pierce and C. E. Wayne, *On the validity of mean-field amplitude equations for counterpropagating wavetrains*, Nonlinearity **8** (1995), no. 5, 769–779.
35. W. Schlag, *Spectral theory and nonlinear partial differential equations: a survey*, Discr. Cont. Dyn. Syst. **15** (2006), no. 3, 703–723.

36. G. Schneider, *Justification of modulation equations for hyperbolic systems via normal forms*, NoDEA, Nonlinear Differ. Equ. Appl. **5** (1998), no. 1, 69–82.
37. G. Schneider, *Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances*, J. Differ. Equations **216** (2005), no. 2, 354–386.
38. G. Schneider and H. Uecker, *Existence and stability of modulating pulse solutions in Maxwell's equations describing nonlinear optics*, Z. Angew. Math. Phys. **54** (2003), no. 4, 677–712.
39. C. Sulem and P.-L. Sulem, *The Nonlinear Schrodinger Equation*, Springer, 1999.
40. A. Soffer and M. I. Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math. **136** (1999), no. 1, 9–74.
41. G. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, 1974.